In this lecture we will classify all Cohen-Macaulay bipartite graphs.

**Definition 1.** A finite graph \( G \) on the vertex set \([n]\) is bipartite if there exists a partition \([n] = V \cup V', V \cap V' = \emptyset\) such that each edge of \( G \) is of the form \( \{i, j\} \) with \( i \in V \) and \( j \in V' \).

A graph \( G \) is bipartite if it has no cycles of odd length.

The following figure gives an example of a bipartite graph.

![Figure 1:]

We say a graph \( G \) is a Cohen-Macaulay graph if \( S/I(G) \) is Cohen-Macaulay.

**Theorem 2 (Herzog-Hibi).** Let \( G \) be a bipartite graph with vertex partition \( V \cup V' \). Then the following conditions are equivalent:

(a) \( G \) is a Cohen-Macaulay graph;

(b) \( |V| = |V'| \) and the vertices \( V = \{x_1, \ldots, x_n\} \) and \( V' = \{y_1, \ldots, y_n\} \) can be labelled such that:

(i) \( \{x_i, y_i\} \) are edges for \( i = 1, \ldots, n \);

(ii) if \( \{x_i, y_j\} \) is an edge, then \( i \leq j \);

(iii) if \( \{x_i, y_j\} \) and \( \{x_j, y_k\} \) are edges, then \( \{x_i, y_k\} \) is an edge.

---

*This lecture was held by Jürgen Herzog  
University of Essen, Germany  
e-mail: juergen.herzog@uni-essen.de*
Proof. (a) \(\Rightarrow\) (b): If \(G\) is Cohen-Macaulay, then all minimal prime ideals of \(I(G)\) have the same height. If \(V = \{x_1, \ldots, x_n\}\), and \(V' = \{y_1, \ldots, y_m\}\), then \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_m)\) are minimal prime ideals of \(I(G)\). Therefore, \(n = m\).

(i) Let \(U \subset V\), and \(N(U) = \{y_j \in V' : \{x_i, y_j\} \in E(G) \text{ for some } x_i \in U\}\). The well-known marriage theorem in graph theory says: if \(|U| \leq |N(U)|\) for all subsets \(U\) of \(V\), then the vertices can be labelled such that \(\{x_i, y_j\} \in E(G)\) for all \(x_i \in V\).

Now \((V \setminus U) \cup N(U)\) is a vertex cover for all \(U\). Since \(I(G)\) is unmixed, all minimal vertex covers have size \(n\). So
\[
n - |U| + |N(U)| \geq |(V \setminus U) \cup N(U)| \geq n.
\]
Hence \(|N(U)| \geq |U|\), and the marriage theorem may be applied.

(ii) Let \(\Delta\) be the simplicial complex with \(I_\Delta = I(G)\). Then \(V\) and \(V'\) are facets of \(\Delta\), while \(\{x_i, y_j\}\) (with the labelling as above) are not faces of \(\Delta\).

Since \(\Delta\) is Cohen-Macaulay it is strongly connected. In particular, there exist facets
\[
V = F_0, \ldots, F_n = V'
\]
such that \(F_i\) and \(F_{i-1}\) intersect in codimension 1 for \(i = 1, \ldots, n-1\). Then \(|F_1 \setminus F_0| = 1\), say \(F_1 \setminus F_0 = \{y_1\}\). Therefore \(F_1 = \{y_1, x_2, \ldots, x_n\}\) because \(\{x_1, y_1\}\) is not a face.

Similarly, \(|F_2 \setminus F_1| = 1\), say \(F_2 \setminus F_1 = \{y_2\}\). Therefore \(F_2 = \{y_1, y_2, x_3, \ldots, x_n\}\) because \(\{x_2, y_2\}\) is not a face.

Hence by induction we may assume that \(F_i = \{y_1, \ldots, y_i, x_{i+1}, \ldots, x_n\}\) for \(i = 0, \ldots, n\).

In particular, if \(j < i\), then \(\{x_i, y_j\}\) is a face of \(\Delta\), and hence not an edge of \(G\).

(iii) Now we assume that the vertices are labelled such that (i) and (ii) hold.

Let \(\{x_i, y_j\}\) and \(\{x_j, y_k\}\) be edges of \(G\) with \(i < j < k\), and suppose that \(\{x_i, y_k\} \notin E(G)\).

Since \(\{x_i, y_k\}\) is a face of \(\Delta\) and since \(\Delta\) is pure, there exists a facet of \(\Delta\), with \(|F| = n\) and \(\{x_i, y_k\} \subset F\). Since \(F\) is a face of \(\Delta\), any 2-element subsets of \(F\) are non-edges of \(G\).

We have \(y_j \notin F\) since \(\{x_i, y_j\} \in E(G)\) and \(x_i \in F\), and \(x_j \notin F\) since \(\{x_j, y_k\} \in E(G)\) and \(y_k \notin F\).

On the other hand, since \(\{x_i, y_k\} \in E(G)\) for all \(\ell\), the face \(\ell\) never contains both \(x_\ell\) and \(y_\ell\), i.e., \(\ell = \{x_{i_1}, \ldots, y_{i_k}, y_{j_1}, \ldots, y_{j_{n-k}}\}\) with \(\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = [n]\). So either \(y_j \in F\) or \(x_j \in F\), a contradiction. \(\square\)

For the proof of the other direction we make a detour. Let \(L\) be a distributive lattice, i.e., a lattice where join \(p \lor q\) and meet \(p \land q\) satisfy the distributive law.

Given a finite poset \(P\). A subset \(I \subset P\) is a poset ideal, if with each \(p \in I\) all \(q\) with \(q \leq p\) belong to \(I\).

The set \(J(P)\) of poset ideals with union as join, and intersection as meet is a distributive lattice. For example, if the elements in \(P\) are pairwise incomparable, then \(J(P)\) is a Boolean lattice of rank \(|P|\).
The fundamental theorem of Birkhoff says: Given a finite lattice $L$. Let $P \subseteq L$ be the subposet of join irreducible elements. Then

$$L \cong \mathcal{J}(P).$$

Figure 3 identifies the above lattice as the ideal lattice of a poset.

Now for any distributive lattice $L = \mathcal{J}(P)$ we associate an ideal $H_L \subseteq K[\{x_p, y_p\}_{p \in P}]$ as follows: for each poset ideal $I \subseteq P$ let

$$u_I = \prod_{p \in I} x_p \prod_{q \notin I} y_q.$$ 

In our example

$$H_L = (uvwx, avwx, bvwx, abwx, abcx, bdwx, abcd).$$

Here we used the variables $a, b, c, d, u, v, w, x$ instead of $x_1, \ldots, x_4, y_1, \ldots, y_4$. 
Theorem 3. (a) \( H_L \) has a linear resolution.
(b) Let \( b_i = \text{number of intervals } [p, q] = \{ a \in L : p \leq a \leq q \} \) which are isomorphic to a Boolean lattice of rank \( i \). Then \( \beta_i(H_L) = b_i \).

In our example we have \( b_0 = 8, b_1 = 10, b_2 = 3 \).

Corollary 4. \( \sum (-1)^i b_i = 1 \).

Proposition 5. \( H_L = \bigcap_{p \leq q} (x_p, y_q) \).

**Proof.** Let \( (x_p, y_q) \) with \( p \leq q \), and let \( u_I \in H_L \). If \( p \in I \), then \( x_p | u_I \), and hence \( u_I \in (x_p, y_q) \). If \( p \notin I \), then \( q \notin I \), and hence \( y_q | u_I \), that is, \( u_I \in (x_p, y_q) \). Therefore, \( H_L \subset \bigcap_{p < q} (x_p, y_q) \).

Conversely, let \( p \) be a monomial prime ideal with \( H_L \subset p \). If \( p \) contains no \( x_p \), then \( u_p = \prod_{p < q} x_p \notin p \), a contradiction. If \( p \) contains no \( y_q \), then \( u_0 = \prod_{q < p} y_q \notin p \), a contradiction. Therefore \( x_p \in p \) for some \( p \in P \), and \( y_q \in p \) for some \( q \in P \). Let \( p \) be minimal with this property. We show that \( y_q \in p \) for some \( q \geq p \). (Then \( (x_p, y_q) \subset p \), as desired).

Suppose that for all \( y_q \in p \) we have \( q < p \). Let \( I \) be the poset ideal \( \{ r \in P : r < p \} \). By the choice of \( x_p \in p \) we have that \( x_r \) does not divide \( u_I \) for any \( x_r \in p \). Also \( y_q \) does not divide \( u_I \) for any \( y_q \in p \) since \( q \in I \). Hence \( u_I \notin p \), a contradiction. \( \square \)

Let \( \Delta \) be the simplicial complex with \( I_\Delta = H_L \), \( L = J(P) \). Then by Lemma 18 in [8] and Proposition 5 we have

\[ I_{\Delta'} = (x_p y_q)_{p \leq q}. \]

Since \( H_L \) has a linear resolution, \( S/I_{\Delta'} \) is Cohen-Macaulay, and it is the edge ideal of a bipartite graph, which we denote by \( G(P) \).

So for each finite poset \( P \), \( G(P) \) is a Cohen-Macaulay bipartite graph (satisfying the conditions (b), (i), (ii) and (iii) of Theorem 2).

Now we can prove Theorem 2 (b) \( \Rightarrow \) (a): Let \( G \) be bipartite graph satisfying the conditions (b) of Theorem 2 with vertex decomposition

\[ V = \{ x_p \}_{p \in P} \quad \text{and} \quad V' = \{ y_p \}_{p \in P}. \]

where \( P = [n] \).

On \( P \) we define a partial order \( < \) by setting \( p < q \) if \( \{ x_p, y_q \} \in E(G) \). Then \( G = G(P) \) and hence \( G \) is Cohen-Macaulay.

We conclude this lecture with an example of a bipartite graph which satisfies the conditions of Theorem 2, and hence is Cohen-Macaulay.
COHEN-MACAUMLY BIPARTITE GRAPHS

Figure 4:

References


