GENERIC INITIAL IDEALS *

Let $K$ be a (commutative) infinite field, $r \geq 1$, $R = K[X_1, X_2, \ldots, X_r]$ and “$<$” be an order relation on the set of all monomials of $R$. We assume that $X_1 > X_2 > \ldots > X_r$, “$<$” is multiplicative (i.e. $m > m'$ implies $mn > m'n$, for any monomials $m, m', n$) and “$<$” is degree-sensitive, that is $\text{deg}(m) > \text{deg}(m')$ implies $m > m'$, for any monomials $m, m'$. If $I$ is a homogeneous ideal of $R$, we consider the initial ideal of $I$, that is the ideal of $R$ generated by the initial monomial, $\text{in}(f)$, of all nonzero elements $f$ of $I$. This is obviously a monomial ideal that we shall denote by $\text{in}(I)$.

The concept of Generic initial ideal (abbrev. GIN) was introduced by R. Hartshorne in his paper: *Connectness of Hilbert schemes*, Publ. IHES 29(1966), 5-48. The power series analogue of GIN was introduced by H. Grauert in his paper: *Über die Deformation isolierter Singularitäten analytischer Mengen*, Invent. Math. 15(1972), 171-198.

$\text{GL}_r(K)$ acts as a group of automorphisms of the graded ring $R$, by

$$g(X_j) = \sum g_{ij} X_j, \quad g(X_1^{a_1} \ldots X_r^{a_r}) = g(X_1)^{a_1} \ldots g(X_r)^{a_r}, \quad \text{where} \quad g = (g_{ij}).$$

In $\text{GL}_r(K)$ we distinguish the subgroup $B$ consisting of all nonsingular upper triangular matrices, also called the Borel subgroup. We notice that the set of all diagonal matrices and all upper elementary matrices generates $B$.

Let $R_d$ be the $K$-vector space consisting of all forms of degree $d$. If $V$ is a $t$-dimensional subspace of $R_d$ and $f_1, \ldots, f_t$ is a $K$-basis of $V$, then the $t$-exterior power $\wedge V$ of $V$ equals $K f_1 \wedge f_2 \wedge \ldots \wedge f_t$. If $f_1, f_2, \ldots, f_t$ and $g_1, g_2, \ldots, g_t$ are linearly independent systems in $R_d$, then they generate the same $K$-subspace if and only if $f_1 \wedge f_2 \wedge \ldots \wedge f_t = \lambda g_1 \wedge g_2 \wedge \ldots \wedge g_t$ for some nonzero $\lambda$ in $K$. If $m_1, m_2, \ldots, m_t$ are distinct monomials of $R_d$, then $m_1 \wedge m_2 \wedge \ldots \wedge m_t$ will be called monomial of $\wedge^t R_d$. Such a monomial $m_1 \wedge m_2 \wedge \ldots \wedge m_t$ will be called normal if $m_1 > m_2 > \ldots m_t$. On the set of all normal monomials of $\wedge^t R_d$ we introduce the order relation given by $m_1 \wedge m_2 \wedge \ldots \wedge m_t > m_1' \wedge m_2' \wedge \ldots \wedge m_t'$ if and only if $(m_1, m_2, \ldots, m_t) > (m_1', m_2', \ldots, m_t')$ in the lexicographic order. Then for each $f$ in $\wedge^t R_d$, we can consider its initial monomial $\text{in}(f)$. For instance, if $f_1, f_2, \ldots, f_t$ is basis of $\wedge^t R_d$.

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V with \( \text{in}(f_1) > \text{in}(f_2) > \ldots \text{in}(f_t) \) (note that any \( V \) has such a basis), then \( \text{in}(f_1 \land f_2 \land \ldots \land f_t) = \text{in}(f_1) \land \text{in}(f_2) \land \ldots \land \text{in}(f_t) \).

The following remark is now obvious.

**Remark 1.** Let \( I \) be a homogeneous ideal of \( R \) and \( f_1, f_2, \ldots, f_t \) a \( K \)-basis of \( I_d \), the component of degree \( d \) of \( I \). Then the monomial basis of \( \text{in}(I_d) \) is given by \( \text{in}(f_1 \land f_2 \land \ldots \land f_t) \). In particular, if \( g \in GL_r(K) \), the monomial basis of \( \text{in}(gI_d) \) is given by \( \text{in}(g(f_1) \land g(f_2) \land \ldots \land g(f_t)) \).

We have the following theorem of existence of GIN.

**Theorem 2.** Let \( K \) be an infinite field and \( I \) a nonzero homogeneous ideal of \( R \). There exists a nonempty open Zariski set \( U \) of \( GL_r(K) \), such that \( \text{in}(gI) = \text{in}(g'I) \) for each \( g, g' \in U \) (this constant ideal is called the generic initial ideal of \( I \) and is denoted by \( \text{Gin}(I) \)). Moreover, if \( J = \text{Gin}(I) \) and \( I_d \) is \( t \)-dimensional over \( K \), then \( \text{Gin}(I) \) is generated by \( \max \{ \text{in}(f); \ f \in \text{Gin}(I) \} \).

**Proof.** Let \( f_1, f_2, \ldots, f_t \) be a basis of \( I_d \) and \( f = f_1 \land f_2 \land \ldots \land f_t \). Let \( h = (h_{ij}) \) be an \( r \times r \) generic matrix and \( h(f) = h(f_1) \land h(f_2) \land \ldots \land h(f_t) \). Let \( \text{in}(h(f)) = m_1 \land m_2 \land \ldots \land m_t \) and \( p(h) \) its coefficient from \( K[h_{11}, \ldots, h_{rr}] \). By our previous remark, \( \text{in}(gI_d) \) is \( K \)-generated by \( m_1, \ldots, m_t \) if and only if \( p(g) \neq 0 \). Denote by \( U_d \) the set of all \( g \in GL_r(K) \) with this property. Thus, for each \( d \geq 1 \) we have defined the nonempty open subset \( U_d \) of \( GL_r(K) \) and the \( K \)-vector subspace \( J_d = Km_1 + \ldots + Km_t \). Let \( J = \sum_{d \geq 1} J_d \). We claim that \( J \) is an ideal of \( R \). Indeed, since \( R_e = R_{e+1}^* \), it suffices to see that \( R_1 J_d \subseteq J_{d+1} \). Since \( U_d \) is dense in \( GL_r(K) \), there exists \( g_0 \in U_d \cap U_{d+1} \). Then

\[
R_1 J_d = R_1 \text{in}(g_0 I_d) \subseteq \text{in}(g_0 I)_{d+1} = J_{d+1}.
\]

Also, \( J_d \) and \( \text{in}(gI_d) \) have the same dimension over \( K \), for each \( g \in GL_r(K) \). Indeed,

\[
dim_K(J_d) = \dim_K(\text{in}(g_0 I_d)) = \dim_K((g_0 I)_{d+1}) = \dim_K(I_{d+1}) \text{ and } \dim_K(I_d) = \dim_K(in(gI_d)).
\]

\( J \) has a finite system of generators \( v_1, v_2, \ldots, v_q \) consisting of homogeneous polynomials and let \( e \) be the maximum of the degrees of these polynomials. If \( g \in U = U_1 \cap \ldots \cap U_e \), we claim that \( J_d' \subseteq \text{in}(gI_d') \) for each \( d' \). This fact is clear for \( d' \leq e \) and then follows by induction, so we may assume that \( d' = e + 1 \). In this case, \( J_{e+1} = R_1 J_e \subseteq R_1 \text{in}(gI) \subseteq \text{in}(gI)_{e+1} \). By equidimensionality, \( J_d = \text{in}(gI_d) \), for each \( d \geq 1 \) and \( g \in U \). In particular, \( U \) is the desired Zariski open set. \( \Box \)
The following result is due to Galligo (see Lecture Notes in Math. 409, Springer Verlag 1974) and Bayer-Stillmann (Duke Math. J. 55(1987), 321-328).

**Theorem 3.** $\text{Gin}(I)$ is a Borel-fixed ideal, that is $g\text{Gin}(I) = \text{Gin}(I)$ for each $g \in B$.

**Proof.** Replacing $I$ by some $gI$, we may assume $\text{Gin}(I) = \text{in}(I)$. Let $1 \leq i < j \leq r$ and let $\gamma \in \text{GL}_r(K)$ be the matrix obtained from the identity matrix adding the $i$th column to the $j$th one. Since all the matrices of this kind generate $B$, it suffices to see that $\gamma \text{Gin}(I) \subseteq \text{Gin}(I)$. Let $f_1, \ldots, f_t$ be a $K$-basis of $I_d$ with $\text{in}(f_1) > \text{in}(f_2) > \ldots \text{in}(f_t)$ and denote $\text{in}(f_i)$ by $m_i$. Suppose that $\gamma(\text{in}(I_d)) \not\subseteq \text{in}(I_d)$. This means that $\gamma(\text{in}(f)) \neq \text{in}(f)$, where $f = f_1 \wedge f_2 \wedge \ldots \wedge f_t$. Also

$$
\gamma(m_1 \wedge m_2 \wedge \ldots \wedge m_t) = \gamma(m_1) \wedge \gamma(m_2) \wedge \ldots \wedge \gamma(m_t).
$$

Due to the action of $\gamma$, in the expression of $\gamma(\text{in}(f))$ appears a monomial $n > m$. Let us associate to each monomial $\rho_1 \wedge \rho_2 \wedge \ldots \wedge \rho_t \in \Lambda^t \text{R}_d$, its weight $\rho_1 \rho_2 \ldots \rho_t$. When we express $f$ as a sum of monomials, $m(f)$ is the only monomial of maximal weight, say $w_0$. Then $f$ can be written as

$$
f = f_{w_0} + \sum_w f_w,
$$

where $f_{w_0} = \text{in}(f)$ and $f_w$ is the some of all monomials in $f$ of weight $w$. So, if $\delta$ is the diagonal matrix given by $\delta_1, \delta_2, \ldots, \delta_r$, it follows that

$$
\delta(f) = w_0(\delta_1, \ldots, \delta_r)\text{in}(f) + \sum_{w \geq w_0} w(\delta_1, \ldots, \delta_r)f_w,
$$

with $w_0(\delta_1, \ldots, \delta_r)$ and $w(\delta_1, \ldots, \delta_r)$ in $K$. So

$$
(\gamma \delta)(f) = w_0(\delta_1, \ldots, \delta_r)\gamma(\text{in}(f)) + \sum_{w \geq w_0} w(\delta_1, \ldots, \delta_r)\gamma(f_w).
$$

Then $n$ appears in the expression of $\gamma \delta(f)$ with the coefficient

$$
aw_0(\delta_1, \ldots, \delta_r) + \sum_{w \in W} a_w w(\delta_1, \ldots, \delta_r),
$$

where $W$ is the set of those $w$ such that $n$ appears in $\gamma(f_w)$. But the polynomial

$$
P(X_1, X_2, \ldots, X_r) = aw_0(X_1, \ldots, X_r) + \sum_{w \in W} a_w w(X_1, \ldots, X_r)
$$

is nonzero and $K$ is infinite, so $P(\delta_1, \ldots, \delta_r) \neq 0$ for some $\delta_1, \ldots, \delta_r \in K$. This way, we contradict the last assertion of the preceding theorem. \qed
Corollary 4. If $K$ has characteristic zero, then $\text{Gin}(I)$ is a strongly stable ideal, that is, whenever a monomial $m$ belongs to $\text{Gin}(I)$, $X_j$ divides $m$ and $i < j$, then $X_i(m/X_j)$ is also in $\text{Gin}(I)$.

Proof. Let $J = \text{Gin}(I)$, $m$ a monomial of $J$, $X_j$ divide $m$ and $i < j$. Let $m = X_1^{a_1} \ldots X_i^{a_i} \ldots X_j^{a_j} \ldots X_r^{a_r}$ and $b$ the element of $B$ obtained by making the $(i,j)$ entry of the identity matrix equal to 1. Since $\text{Gin}(I)$ is Borel-fixed, $bm \in J$. But

$$bm = X_1^{a_1} \ldots X_i^{a_i} \ldots (X_j + X_i)^{a_j} \ldots X_r^{a_r} = m + a_j X_i(m/X_j) + \ldots,$$

so $a_j X_i(m/X_j) \in J$ because $J$ is monomial, thus $X_i(m/X_j) \in J$ because $a_j \neq 0$. 

\qed