Contributions to the study of the connections between algebraic hyperstructures and fuzzy sets

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Introduction

In 1934, at the 8th Congress of Scandinavian Mathematicians, F. Marty has introduced, for the first time, the notion of hypergroup, using it in different contexts: algebraic functions, rational fractions, non-commutative groups. This moment was the first step in the history of the development of the Algebraic Hyperstructures Theory all over the world, especially in Europe (France, Italy, Greece, Romania, Czeck and Slovak Republics, Montenegro, Serbia, Finland, Holland, Germany), Australia, S.U.A. and Canada, and later also in Iran, China, Thailand, Japan, Korea, Jordan. At the beginning, the general aspects of the theory, the connections with groups and various applications in geometry (redefining a spherical, a descriptive or a projective geometry using hypergroups) have been studied. The theory knew an important progress starting with the 70's, when its research area has enlarged. In France, M.Krasner, M.Koskas and Y.Sureau have investigated the theory of subhypergroups and the relations defined on hyperstructures; in Greece, J.Mittas, D.Stratigopoulos, M.Konstantinidou, K.Serafimidis, Ch.G.Massouros have studied the canonical hypergroups, the hyper-rings and the hyper-modules, the hyper-lattices, the hyper-fields with applications in Automata Theory. T.Vougiouklis has analyzed especially the cyclic hypergroups, the P-hyperstructures and the representation of the H_ϑ-structures. Significant contributions in the study of the regular hypergroups, the complete hypergroups, of the heart and of the homomorphisms of hypergroups in general or with applications in Combinatorics and Geometry have been brought by the group of Italian mathematicians represented by P.Corsini, M.De Salvo, R.Migliorato, F.De Maria, G.Romeo, P.Bonansinga, D.Freni.

Beginning with the '90s, Hyperstructures Theory represents a constant concern also for the Romanian mathematicians, the decisive moment being the 5th International Congress of Hyperstructures and Applications, organized in 1993, at the University "Al.I.Cuza" of Iasi. This domain of the modern algebra is a topic of a great interest also for the Romanian researchers, who published a lot of papers on hyperstructures in national and international journals, gave communications in conferences and congresses, wrote PhD theses in this field. We quote the papers written by M.Ştefănescu, I.Tofan, Gh.Radu, V.Leoreanu, C.Volf, M.Gontineac, C.Pelea, M. and C.Guţan, M. Tărnăuceanu, I.Cristea.

Till now applications of the hypergroups in various domains, as: geometry, topology, analysis of the convex systems, finite groups' character theory, cryptography and codes theory, automata theory, graphs and hypergraphs, theory of fuzzy and rough sets, theory of binary relations, probability theory, ethnology, economy, etc, have been founded.

The **first chapter** of this thesis contains the presentation of some notions and results in the hypergroup theory, necessary later on. It is now well-known that the hypergroup theory is a natural generalization of the group theory, applied in informatics. In hypergroups, identities and the inverses, as well as the notions of subhypergroups, homomorphism of hypergroups, relations on hypergroups, factor hypergroup are defined naturally. The properties of the hyperproduct allow us to consider different types of subhypergroups: closed, ultraclosed, invertible, complete parts. We recall the definition of the fundamental relation β, which helps us to determine the heart of a hypergroup, the subhypergroup which offers many information about the structure of a given hypergroup. Our presentation goes on by introducing some particular hypergroups, the studying their properties, their heart and determining connections among these classes of hypergroups. So we consider the regular hypergroups, the reversible hypergroups, the canonical, the complete ones, the join spaces.

In the end of the first chapter we recall some results from the join spaces theory, established by Prenowitz and Jantosciak in [43], used prioritarily in the **second chapter**, where we show how to reconstruct, from an algebraic point of view, three branches of the geometry: the projective, the descriptive and the spherical one. Each of them can be characterized by a set S of points and a ternary relation (abc), read "b is between a and c", which verifies some axioms. Using, if necessarily, an ideal point $e \notin S$, Prenowitz and Jantosciak have defined on the set S, in the descriptive geometry, and respectively on the new set $S' = S \cup \{e\}$, in the other two geometries, a join hyperoperation (Theorem 2.2 and 2.3, Theorem 2.5 and 2.6, Theorem 2.13), succeeding in characterizing the given geometries as particular join spaces.

In the **Chapter** 3 we describe three types of connections between the hypergroups and the fuzzy sets, which lead us to obtain new join spaces. The first relationship, considered by Rosenfeld in 1971, has been extended by Ameri and Zahedi [1] in 1997. Given a group G and a fuzzy subset μ of it, one defines the hyperoperation induced by μ:

$$\circ_\mu : G \times G \longrightarrow \mathcal{P}^*(G), a \circ_\mu \ b = \mu^a \mu^b.$$

The Theorem 3.1.6 determines conditions under which the new hyperstructure is a hypergroup, a canonical hypergroup or a join space.

In the subsection 3.2 some results obtained by Corsini and Leoreanu in [9], [10], [11], [13], [14] about the join space obtained when one endows a nonempty set H with the following hyperproduct:

$$x \circ y = \{z \in H \mid \min\{\mu(x), \mu(y)\} \leq \mu(z) \leq \max\{\mu(x), \mu(y)\}\}$$

are recalled; here μ is a fuzzy subset of H. Necessary and sufficient conditions that two such join spaces associated with two different fuzzy subsets defined on the same universe H are isomorphic are presented (Theorem 3.2.3 and Theorem 3.2.4).

The third connection between the hypergroups and the fuzzy sets has been defined by Corsini in [17] and studied then by P.Corsini, M.Ştefănescu, V.Leoreanu, I.Cristea. With a hypergroupoid $\langle H, \circ \rangle$, one associates the fuzzy set $\widetilde{\mu}$ by the relations (ω) from § 3.3.1 and then the join space $\langle {}^1H, \circ_1 \rangle$, as in the previous association; using again the construction (ω) for the new hypergroupoid $\langle {}^1H, \circ_1 \rangle$, one determines the fuzzy set $\widetilde{\mu}_1$ and the corresponding join space $\langle {}^2H, \circ_2 \rangle$ and so on. The process stops when one obtains two successively equal or isomorphic join spaces. In this way one constructs a sequence of join spaces and fuzzy sets $(\langle {}^iH, \circ_i \rangle, \widetilde{\mu}_i)_{i\geq 1}$ associated with H; its length is called the fuzzy grade of the hypergroupoid H.
For any $i \geq 1$, there exist $r \in \mathbb{N}^*$ and a partition $\pi = \{{}^iC_j\}_{j=1}^r$ of ${}^iH$ such that $x, y \in {}^iC_j \Longleftrightarrow \widetilde{\mu}_{i-1}(x) = \widetilde{\mu}_{i-1}(y)$. Denoting $k_{j_l} = \mid {}^iC_{j_l} \mid$, $1 \leq l \leq r$, with every join space ${}^iH$ one associates an ordered r-tuple $(k_{j_1}, k_{j_2}, ..., k_{j_r})$. Particularly, with any hypergroupoid H we associate an ordered r-tuple $(k_1, k_2, ..., k_r)$. Studying certain types of ordered r-tuples, we obtain some properties of the fuzzy grade of a hypergroupoid (see § 3.3.2.).

If with the finite hypergroupoid $H = \{a_1, a_2, ..., a_n\}$ we associate the n-tuple $(1, 1, ..., 1)$, we can easily construct, depending on the parity of n and without calculating the effective values $\widetilde{\mu}_1(a_i)$ using the relations (ω), the associated join space ${}^1H$. Moreover, if $n + 1 = 2^s$, we prove that $s.f.g.(H) = s + 1$ (see the Remark 3.3.11). An immediate consequence of this theorem is given by the Corollary 3.3.14: Given a natural number $n \in \mathbb{N}^* \setminus \{1\}$ one can construct a join space H with the fuzzy grade equal to n. The Theorem 3.3.15 expresses a sufficient condition, but not a necessary one, that two successively join spaces of the sequence are not isomorphic: if with the join space ${}^iH$ we associate the r-tuple $(k_1, k_2, ..., k_r)$ with the property $(k_1, k_2, ..., k_r) = (k_r, k_{r-1}, ..., k_1)$, then ${}^iH$ and ${}^{i+1}H$ are not isomorphic.

One of the class of hypergroups studied in the following is that of the finite complete hypergroups, that is the one of the hypergroups H which can be represented as $H = \bigcup\limits_{g\in G} A_g$, where

1) $(G, \cdot)$ is a group;

2) for any $(g_1, g_2) \in G^2, g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$;

3) if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.

We have shown that, for a complete hypergroup H of order n, its fuzzy grade depends only on the m-decomposition of n. The Theorem 3.3.20 determines all the complete hypergroups of order smaller than 7 and their fuzzy grade. Moreover, any complete hypergroup of order n is characterized by a m-tuple $[k_1, k_2, ..., k_m]$, where $m = |G|$, $2 \leq m \leq n - 1$, $G = \{g_1, g_2, ..., g_m\}$, and for any $i \in \{1, 2, ..., m\}$, $k_i = |A_{g_i}|$.
If H is a complete hypergroup of the type $[\underbrace{p, p, ..., p}_{k \quad times}, kp]$, where $n = |H| = 2kp$,

then $s.f.g.(H)$=2 (Proposition 3.3.21); if it is of the type $[\underbrace{p,p,...,p}_{s\ times},\underbrace{k,k,...,k}_{t\ times},ps]$, with $2 \leq p < k < ps$, $n = |H| = 2ps + kt$, then for $n = 4ps$, $s.f.g.(H) = 3$, and for $kt \neq 2ps$, $f.g.(H) = 2$ (Proposition 3.3.22); more general, if H is of the type $[\underbrace{k,k,...,k}_{l\ times},\underbrace{p,p,...,p}_{s\ times},\underbrace{s,s,...,s}_{p\ times},\underbrace{l,l,...,l}_{k\ times}]$, with $2 \leq k < p < s < l$, $n = |H| = 2(ps + kl)$, we obtain that, for $kl = ps$, $s.f.g.(H) = 3$ and for $kl \neq ps$, $f.g.(H) = 2$ (Remark 3.3.25).

Next we gave an example of a non-complete 1-hypergroup of order n and we calculated its fuzzy grade depending on the value of n.

Another class of hypergroups is that of the hypergroups with partial scalar identities (i.p.s.hypergroups). We determine the fuzzy grade of all the i.p.s.hypergroups of order smaller than 8 (Theorem 3.3.26).

Given an equivalence relation R on a universe H, one defines a rough set $(\underline{R}(A), \overline{R}(A))$, where A is a nonempty subset of H, and

$$\underline{R}(A) = \bigcup_{R(x)\subset A} R(x) \quad \text{and} \quad \overline{R}(A) = \bigcup_{R(x)\cap A\neq\emptyset} R(x).$$

It is well known that every rough set is a particular fuzzy set, but conversely not ([2]).

Defining on H the hyperoperation:

$$\forall (x,y) \in H^2, x \circ y = \overline{R}(\{x,y\}) \setminus \underline{R}(\{x,y\}),$$

the hypergroupoid $\langle H, \circ\rangle$ is a join space if and only if, for any $x \in H$, $|R(x)| \geq 3$. Attaching with this join space the fuzzy set $\widetilde{\mu}$ defined by the relations (ω), we have obtained that, for any two distinct elements $u, v \in H$, the following implications hold:

(i) $\widetilde{\mu}(u) = \widetilde{\mu}(v) \iff |R(u)|=|R(v)|$;

(ii) $\widetilde{\mu}(u) < \widetilde{\mu}(v) \iff |R(u)|>|R(v)|$.

If R is an equivalence relation on H such that $\mid R(x) \mid= k = const$, for any $x \in H$, then $s.f.g.(H) = 1$.

In the last part of the thesis we give some properties of the join spaces ${}^iH$ associated with a hypergroupoid H, in connection with the fundamental relations defined on H. Besides the fundamental relation β defined on a hypergroupoid H, there exist another three types of equivalence relations, called also fundamental by Jantosciak in [29]: operational equivalence, inseparability and essential indistinguishability. By using them, J.Jantosciak has introduced the notion of reduced hypergroup. We show that, for any hypergroupoid H and any $i \geq 1$, the quotient hypergroup ${}^iH/R_{\widetilde{\mu}_i}$ is reduced (Theorem 3.3.33). This property is not valid for the associated join spaces ${}^iH$, more clearly, the join space ${}^1H$ is the unique join space from the sequence $(\langle {}^iH, \circ_i\rangle, \widetilde{\mu}_i)_{i\geq 1}$ corresponding to H which can be a reduced hypergroup, this happens if and only if the n-tuple associated with H is of the form $(1, 1, ..., 1)$.

In the last section we find some properties of the three fundamental relations and we give examples of remarkable hypergroups which are reduced or not.

In conclusion, this thesis is focused on the three types of connections established till now between the hypergroups and the fuzzy sets, studying especially the sequence of join spaces obtained with this method. We consider some open research directions and we present some problems, a few of them remaining still non-resolved; they are enumerated in the **Chapter** 4 of the thesis.

Acknowledgements

Besides the efforts I put in writing up this thesis, many other people are involved in this project and have contributed to its success in various direct and indirect ways. I appreciate their help and I am glad I had the opportunity to learn from them and work with them.

First of all, I would like to express my deep and sincere gratitude to Professor Mirela Ştefănescu, for her helpful suggestions and discussions along the years of my PhD courses. She encourage me all the time and she took care that I stay on the right track.

I am deeply grateful to Professor Piergiulio Corsini, for his detailed and constructive comments, for his important support throughout this work, during my staying to Udine's University. He is the person who introduced me to the field of hyperstructures and in the world of research.

I wish to express my thanks to Professor Ioan Tofan who encouraged me to start the PhD programme and who gave me the opportunity to earn a scholarship for six months to the University of Udine. That was the moment when I started to understand what means to do research.

My sincere thanks are due to Professor Violeta Leoreanu-Fotea, for her advices and the discussions with her, so important for me in that difficult moments.

I am grateful to Mrs. Elena Mocanu, for helping me to study the LaTex programme.

I owe my loving thanks for my parents, my husband Renato and my step-parents; without their understanding and support it would have been impossible for me to finish this work.

Contents

Chapter 1

Fundamental definitions and results on Hypergroups

In this chapter are presented notions and results obtained on hypergroups (see [4]) which will be used during this thesis. More exactly, we define the notion of hypergroup, subhypergroup, the heart of a hypergroup, the homomorphism of hypergroups; we introduce different types of relations on hypergroups, different types of hypergroups, indicating the connections between them; finally are presented some results from the theory of join spaces, useful in the second chapter.

Let H be a nonempty set and $\mathcal{P}^*(H)$ the set of all nonempty subsets of H.

Definition 1.1. A set H endowed with a hyperoperation $\circ : H^2 \longrightarrow \mathcal{P}^*(H)$ is called a *hypergroupoid* . The image of the pair $(a, b) \in H^2$ is denoted by $a \circ b$ and called the *hyperproduct* of a and b.
If A and B are nonempty subsets of H, then $A \circ B = \bigcup_{\substack{a\in A \\ b\in B}} a \circ b$.

Definition 1.2. The hypergroupoid $\langle H, \circ\rangle$ is called *semihypergroup* if the hyperoperation " $\circ$ " is associative. If H satisfies the *reproducibility law* :

$$\forall\, a \in H, H \circ a = a \circ H = H,$$

then H is called *quasihypergroup.*
A *hypergroup* is a semihypergroup which is also a quasihypergroup.

Definition 1.3. Let $\langle H, \circ\rangle$ be a hypergroupoid .

(i) An element $e \in H$ is called an *identity* if, for all $x \in H$, $x \in x \circ e \cap e \circ x$.
An identity e is called *scalar identity* if, for all $x \in H$, $x \circ e = e \circ x = x$.
An identity e is called *partial identity* if, for any $x \in H$, $x \in x \circ e$ or $x \in e \circ x$.

(ii) An element $x' \in H$ is called an *inverse* of $x \in H$ if there is an identity $e \in H$, such that $e \in x \circ x' \cap x' \circ x$.

Notation. For $a, b \in H$, we denote:
$a/b = \{x \mid a \in x \circ b\}$ and $b \backslash a = \{y \mid a \in b \circ y\}$.
If A and B are subsets of a hypergroup H, we denote $A/B = \bigcup_{\substack{a \in A \\ b \in B}} a/b$.

The homomorpfisms of hypergroups represent an important domain of the theory of hypergroups; they have been studied from D.Ore, M.Dresher, M.Krasner, M.Koskas and later from P.Corsini, S.D.Comer, J.Jantosciak, F.Rossi, P.Bonansigna, V.Leoreanu.

Definition 1.4. Let $\langle H, \circ \rangle$ and $\langle K, \star \rangle$ be hypergroupoids and $f : H \longrightarrow K$ an application from H in K. We say:

(i) f is a *homomorphism* if for all $(a, b) \in H^2, f(a \circ b) \subset f(a) \star f(b)$;

(ii) f is a *good homomorphism* if for all $(a, b) \in H^2, f(a \circ b) = f(a) \star f(b)$.

The equivalence relations defined on a hypergroupoid H form a significant part of theory of hypergroups. We present some results in this direction, insisting on the fundamental relation β, with which is defined the heart of a hypergroup, notion studied by P.Corsini, A.Orsatti and especially by Leoreanu in [35], where she determined the structure of the heart of a join space.

Definition 1.5. Let $\langle H, \circ \rangle$ be a hypergroupoid and let ρ be an equivalence relation on H.

(i) We say that ρ is *regular to the right* if the following implication holds:

$$a \rho b \Longrightarrow \forall\, u \in H, \forall\, x \in a \circ u, \exists y \in b \circ u : x \rho y \ \text{ and } \\ \forall\, \bar{y} \in b \circ u, \exists \bar{x} \in a \circ u : \bar{x} \rho \bar{y}.$$

Similarly, the *regularity to the left* can be defined.
We say that ρ is *regular* if it is regular to the right and to the left.

(ii) We say that ρ is *strongly regular to the right* if the following implication holds:

$$a \rho b \Longrightarrow \forall\, u \in H, \forall\, x \in a \circ u, \forall\, y \in b \circ u : x \rho y.$$

Similarly, the *strong regularity to the left* can be defined.
We say that ρ is *strongly regular* if it is strongly regular to the right and to the left.

Definition 1.6. Let $\langle H, \circ \rangle$ be a hypergroupoid. We define the relation β on H as it follows:

$$a \beta b \Longleftrightarrow \exists n \in \mathbb{N}^*, \exists (x_1, x_2, ..., x_n) \in H^n : a \in \prod_{i=1}^{n} x_i \ni b.$$

Notice that β is a reflexive and a symmetric relation on H, but generally, not a transitive one. Let us denote by β^* the transitive closure of β.

Theorem 1.7. *If $\langle H, \circ \rangle$ is a hypergroupoid, then β^* is the smallest strongly regular equivalence on H, with respect to the inclusion. Moreover, if H is a hypergroup, then $\beta^* = \beta$.*

Definition 1.8. For any equivalence relation ρ defined on a hypergroupoid H and for any element $x \in H$, we define the equivalence class of x related to ρ by $\hat{x} = \{y \in H \mid x\rho y\}$; we denote by H/ρ the set of all equivalence classes of elements of H, called the *factor set.*

Theorem 1.9. *Let $\langle H, \circ \rangle$ be a semihypergroup and ρ an equivalence relation on H.*

(i) *If ρ is regular, then H/ρ is a semihypergroup, with respect to the following hyperoperation:*

$$\forall \, (\bar{x}, \bar{y}) \in (H/\rho)^2, \bar{x} \otimes \bar{y} = \{\bar{z} \mid z \in x \circ y\}.$$

(ii) *Conversely, if the hyperoperation "$\otimes$" is well-defined on H/ρ, then ρ is regular.*

Theorem 1.10. *Let $\langle H, \circ \rangle$ be a hypergroup and ρ a strongly regular equivalence relation on H. Then H/ρ is a group.*
In particular, H/β is a group.

Definition 1.11. Let H be a hypergroup, 1 the identity element of the group H/β and $\varphi_H :\longrightarrow H/\beta$ the canonical projection: for any $x \in H$, $\varphi_H(x) = \hat{x}$.
The *heart* of a hypergroup H is the set $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$.

The complete parts, introduced and studied for the first time by M.Koskas, further have been analyzed by P.Corsini and Y.Sureau, especially in the general context of the theory, but M.De Salvo has studied them from a combinatorial point of view.

Definition 1.12. Let $\langle H, \circ \rangle$ be a semihypergroup and A a non empty subset of H. We say that A is a *complete part* of H if

$$\forall \, n \in \mathbb{N}^*, \forall \, (x_1, ..., x_n) \in H^n, \prod_{i=1}^{n} x_i \cap A \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_i \subset A.$$

Proposition 1.13. *Any intersection of complete parts of a hypergroup H is a complete part of H.*

Definition 1.14. If $\langle H, \circ \rangle$ is a semihypergroup and $A \subset H$, $A \neq \emptyset$, then the *complete closure of A in H* is the intersection of all the complete parts of H, which contain A. It will be denoted by $\mathcal{C}(A)$.

Theorem 1.15. *Let $\langle H, \circ \rangle$ be a semihypergroup.*

(i) *The relation* ρ *defined by* $a\rho b \Longleftrightarrow x \in \mathcal{C}(\{y\})$ *is an equivalence relation.*

(ii) *For any* $(a,b) \in H^2$ *we have* $a\rho b \Longleftrightarrow a\beta^* b$

(iii) *If* A *is a non empty subset of* $\langle H, \circ \rangle$ *, then* $\mathcal{C}(A) = \bigcup_{a \in A} \mathcal{C}(a)$.

Definition 1.16. A semihypergroup $\langle H, \circ \rangle$ is called *complete* if, for any $(x,y) \in H^2$, $\mathcal{C}(x \circ y) = x \circ y$.

Theorem 1.17. *A hypergroup* H *is complete if* $H = \bigcup_{g \in G} A_g$*, where*

1) $(G, \cdot)$ *is a group;*

2) *for any* $(g_1, g_2) \in G^2, g_1 \neq g_2$*, we have* $A_{g_1} \cap A_{g_2} = \emptyset$;

3) *if* $(a,b) \in A_{g_1} \times A_{g_2}$*, then* $a \circ b = A_{g_1 g_2}$.

Definition 1.18. Let $\langle H, \circ \rangle$ be a hypergroupoid and K a non empty subset of H. K is called a *subhypergroupoid* of H if $K \circ K \subset K$. A subhypergroupoid K of H is called a *subhypergroup* of H if $\langle K, \circ \rangle$ is a hypergroup.

Definition 1.19. Let $\langle H, \circ \rangle$ be a hypergroup and K a subhypergroup of it. We say that:

(i) K is *closed to the left* in H if, for any $a \in H$ and any $(x,y) \in K^2$, from $x \in a \circ y$ it follows $a \in K$.
Similarly, we can define the notion of *subhypergroup closed to the right.*
K is *closed* in H if it is closed to the left and to the right.

(ii) K is *ultraclosed to the left* in H if, for any $x \in H, K \circ x \cap (H \setminus K) \circ x \neq \emptyset$.
Similarly, we can define the notion of *subhypergroup ultraclosed to the right.*
K is *ultraclosed* in H if it is ultraclosed to the left and to the right.

The regular hypergroups have been introduced by Dresher and Ore in 1938 (see [25]), later they have been investigated by P.Corsini and A.Orsatti, which determined the heart of a regular reversible hypergroup.

Definition 1.20. A hypergroup H is *regular* if it has at least one identity and each element has at least one inverse. A regular hypergroup $\langle H, \circ \rangle$ is called *reversible* if, for any $(x,y,z) \in H^3$, it satisfies the following conditions:

1) if $y \in a \circ x$, then there exists an inverse a' of a, such that $x \in a' \circ y$;

2) if $y \in x \circ a$, then there exists an inverse a'' of a, such that $x \in y \circ a''$.

Theorem 1.21.

(i) *Any complete hypergroup is regular and reversible.*

(ii) *The heart of a complete hypergroup H is formed by all its identities.*

In 1975, R.Roth utilized the canonical hypergroups to prove some theorems from the theory of the characters of finite groups (see [49]). J.R.McMullen and J.F.Price used generalizations of the canonical hypergroups in the harmonic analysis and in the particle physics.

Definition 1.22. We say that a hypergroup H is *canonical* if

(i) it is commutative;

(ii) it has a scalar identity;

(iii) every element has a unique inverse;

(iv) it is reversible.

Definition 1.23. A canonical hypergroup $\langle H, +\rangle$ is called *strongly canonical* if it satisfies the following conditions:

(i) $\forall\,(x,a) \in H^2, x \in x+a \Longrightarrow x = x+a$;

(ii) $(x+y) \cap (z+w) \neq \emptyset \Longrightarrow x+y \subset z+w$ or $z+w \subset x+y$.

An *i.p.s. hypergroup (a hypergroup with partial identities)* $\langle H, +\rangle$ is a canonical hypergroup such that

$$\forall\,(x,a) \in H^2, x \in x+a \Longrightarrow x = x+a.$$

The notion of join space has been introduced by W.Prenowitz and together with J.Jantosciak he reconstructed, from the algebraic point of view, the projective, the spherical and the descriptive geometry. In the second chapter of the thesis we deal with this subject.

Definition 1.24. A commutative hypergroup $\langle H, \circ\rangle$ is called a *join space* if, for any $(a,b,c,d) \in H^4$, the following implication holds:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

If A and B are subsets of a hypergroup H, we denote $A/B = \bigcup\limits_{\substack{a\in A\\ b\in B}} a/b$.

Theorem 1.25.

(i) *A commutative hypergroup is canonical if and only if it is a join space with a scalar identity.*

(ii) *Any complete commutative hypergroup is a join space.*

Examples of hypergroups:

1) Let G be a group; for any $(x, y) \in G^2$, we define $x \circ y = \langle x, y \rangle$ the subgroup of G generated by x and y. Then $\langle G; \circ \rangle$ is a hypergroup.

2) Let A be a non-empty set and for any $x, y \in A$, $x \circ y = A$. The hypergroup $\langle A; \circ \rangle$ is called the total hypergroup on A.

3) Let G be a group and H be a normal subgroup of G. We define on G the hyperoperation $x \circ y = Hxy$ and we obtain that $\langle H, \circ \rangle$ is a hypergroup.

4) Let $\langle G, + \rangle$ be a commutative group, α an element of second order of G. For any $x, y \in G$ we define $x \circ y = \{x + y, x + y + \alpha\}$. We obtain that $\langle G; \circ \rangle$ is a complete hypergroup.

5) Set $H_n = A \bigcup B \bigcup \{e\}$, where $|A| \geq 2 \leq |B|$, $A \cap B = \emptyset$, $e \notin A \bigcup B$. We fix an element $b_1 \in B$ and we define the hyperoperation in H_n: $\forall a \in A, a \circ a = b_1$; $\forall (a_1, a_2) \in A^2, a_1 \neq a_2, a_1 \circ a_2 = B$; $\forall (a, b) \in A \times B$, $a \circ b = b \circ a = e$; $\forall (b, b') \in B^2, b \circ b' = A$; $\forall a \in A, a \circ e = e \circ a = A$; $\forall b \in B, b \circ e = e \circ b = B$; $e \circ e = e$. H_n is a non complete 1-hypergroup.

6) Any commutative hypergroup endowed with an identity is regular.

Further on we present some results from the theory of join spaces (see [43]).

Proposition 1.26. *Any intersection of closed subhypergroups of a hypergroup H is a closed subhypergroup of H.*

Definition 1.27. The *closed subhypergroup generated by* N, denoted $\langle N \rangle$, is the smallest closed subhypergroup of H containing N, that is the intersection of all closed subhypergroups of H containing N.

One introduces a relation of congruence with respect to a closed subhypergroup N of H in order to construct a new join space, the factor space of H and N.

Definition 1.28. Let N be a nonempty closed subhypergroup of a join space H. Let $a, b \in H$. Then $a \equiv b(modN)$, read *a is congruent to b modulo N*, if $aN \cap bN \neq \emptyset$. The relation $\equiv (modN)$ is an equivalence relation in H and $(a)_N$ denotes the equivalence class of a in H, *the coset of N determined by a.*
The relation $\equiv (modN)$ can be extended to a relation on the subsets of H as it follows: $A \equiv B(modN)$ means that for each $a \in A$ there exists $b \in B$ such that $a \equiv b(modN)$ and conversely.

On the factor set $H\!:\!N = \{(x)_N \mid x \in H\}$ formed by all cosets of N determined by elements of H, we define the join of the cosets $(a)_N, (b)_N$ by $(a)_N \odot (b)_N = \{(x)_N \mid x \in a \circ b\}$. Then $(H\!: N, \odot)$ is a join space with identity N, called the *factor space* H modulo N. The *order* of $H\!:\!N$ is the cardinality of the set $H\!:\!N$.

Definition 1.29. If A and B are closed subhypergroups of the join space $\langle H, \circ \rangle$ such that $B \subset A$ and the order of $A : B$ is 3, then we say that B *separates* A.

Definition 1.30. If the join space H has a scalar identity e, we set $E = \{e\}$, otherwise $E = \emptyset$. We define $\lhd \emptyset \rhd = E$ and if $A \in \mathcal{P}^*(H)$, $\lhd A \rhd = \langle A \rangle$.

A join space H is called an *exchange space* if it satisfies the following conditions:

(I) if $a \in \lhd b \rhd, a \notin E$, then $\lhd a \rhd = \lhd b \rhd$;

(II) if $c \in \lhd a, b \rhd$ and $c \notin \lhd b \rhd$, then $\lhd c, b \rhd = \lhd a, b \rhd$.

Theorem 1.31. *Let H be a join space with a scalar identity e. Then H satisfies* (I) *if and only if satisfies* (II).

Definition 1.32. Let B be a subset of a join space H. B is called *independent* if for any $b \in B$ we have $b \notin \langle B \setminus \{b\} \rangle$.

Definition 1.33. A subset B of a closed subhypergroup N of an exchange space H is called a *basis* of N if it is independent and furthermore $\langle B \rangle = N$.

Any closed subhypergroup of an exchange space has a basis.

Any two bases of N have the same cardinal number called the *dimension* of N, denoted by $d(N)$.

If N and M are finite dimensional closed subhypergroups of $\langle H, \circ \rangle$ such that $N \cap M \neq \emptyset$ then the *dimensional equality* holds:

$$d(\langle N, M \rangle) + d(N \cap M) = d(N) + d(M).$$

If N covers M, then $d(N) = d(M) + 1$.

If $d(N) = n$ is a finite number, any independent set of n elements of N represents a basis of N.

Chapter 2

Geometries and Join spaces

One of the major purpose of the hypergroups theory is that to construct new hyperstructures and to find necessary and sufficient conditions for them to become join spaces. A such construction is studied in this thesis and, for this reason, in the Chapter 2 we present the context where the concept of join space appeared.

The notion of join space has been introduced and studied for the first time by W.Prenowitz. Later, together with J.Jantosciak, he has reconstructed, from the algebraic point of view, several branches of geometry: the projective, the descriptive and the spherical geometry (see [28], [43], [44], [45], [46]). All these three geometries are join theoretic in character, in the sense that a central role is played by an operation "join", which assigns to two distinct points an appropriate connective: in descriptive geometry, a segment; in spherical geometry, minor arc of great circle; in projective geometry, a line. Although the three treatments employ similar methods and contain many similar results, the formulations do not seem to permit a uniform development of the three theories.

For the detailed proofs of the following theorems see [44], [45], [46].

1. The projective geometry is a non-Euclidian geometry which has been developed at he beginning of the 19th century, based on a fundamental principle: the parallel lines meet to the infinite. It is the geometry of the properties which remain invariant on the projections.

Definition 2.1. A *projective geometry* is a system (S, T) formed by a set S of points and a set T of lines which verifies the following postulates:

(P1) A line is a set of points and contains at least three points.

(P2) Two distinct points a, b are contained in a unique line, denoted $L(ab)$.

(P3) If a, b, c, d are distinct and $L(ab) \bigcap L(cd) \neq \emptyset$, then $L(ac) \bigcap L(bd) \neq \emptyset$.

Given a projective geometry (S, T), we can associate to it a join space. One constructs a combinatory system $(S', \cdot)$ in the following way.
Let $S' = S \bigcup \{e\}$, where $e \notin S$, called an ideal point.
First, we suppose $T \neq \emptyset$. A join operation " $\cdot$ " is defined in S' as it follows:

1) If $a, b \in S, \quad a \neq b$, then $a \cdot b = L(ab) \setminus \{a, b\}$.

2) Let $a \in S$. If some line of T contains exactly three points, then $a \cdot a = e$; otherwise $a \cdot a = \{a, e\}$.

3) If $a \in S'$, then $e \cdot a = a \cdot e = a$.

For $T = \emptyset$ and $S = \{a\}$, it can be possible to define two operations on S' (for both of them e is an identity): $a \cdot a = \{e\}$ or $a \cdot a = \{e, a\}$.
Finally, if $T = \emptyset$ and $S = \emptyset$ we define $e \cdot e = e$.

Note that $(S', \cdot)$ has the properties:

(i) $e \in a \cdot b$ if and only if $a = b$

(ii) $a/b = a \cdot b$.

Theorem 2.2. *The system $(S', \cdot)$ is a join space with identity e for which the closed subhypergroup $\langle a \rangle, \quad a \neq e$, generated by a has the cardinal number equal to 2. $(S', \cdot)$ is called the associated join space of the projective geometry (S, T).*

Proof. We have to verify that $(S', \cdot)$ has the properties of a join space:

(JS1) $a \cdot b \neq \emptyset$

(JS2) $a \cdot b = b \cdot a$

(JS3) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(JS4) $a/b \bigcap c/d \neq \emptyset$

(JS5) $a/b \neq \emptyset$.

It is obviously that $(S', \cdot)$ satisfies (JS1), (JS2), (JS5).
We proceed to verify (JS4). Suppose in S', $a/b \bigcap c/d \neq \emptyset$. By the previous property (ii), there exists $p \in a \cdot b \bigcap c \cdot d$. We have to consider the situations:

a) the points $a, b, c, d \in S$ are distinct and non collinear;

b) the points $a, b, c, d \in S$ are distinct and collinear;

c) the points $a, b, c, d \in S$ are non distinct;

d) one of the points a, b, c, d is the ideal point e.

We show the case a), the other possibilities are treated similarly. We have $p \in a \cdot b \bigcap c \cdot d \iff p \in (L(ab) \setminus \{a, b\}) \bigcap (L(cd) \setminus \{c, d\})$, so $p \in L(ab) \bigcap L(cd)$. By (P3), $L(ad) \bigcap L(bc) \neq \emptyset$. Therefore $(a \cdot d \bigcup \{a, d\}) \bigcap \, (b \cdot c \bigcup \{b, c\}) \neq \emptyset$.
Suppose $b \in a \cdot d \iff d \in a/b$, that is $p \neq d \in L(ab) \bigcap L(cd)$. By (P2), $L(ab) = L(cd)$ contrary to hypothesis. Thus $b \notin a \cdot d$. Similarly $c \notin a \cdot d$, $a \notin b \cdot c$, $\; d \notin b \cdot c$.
Hence the only possibility is $a \cdot d \bigcap b \cdot c \neq \emptyset$ and (JS4) holds.

Now we verify (JS3). Let $x \in (a \cdot b) \cdot c$, with $a, b, c \in S'$. Then $x/c \bigcap a \cdot b \neq \emptyset$, $x/c \bigcap b/a \neq \emptyset$. By (JS4), $x \cdot a \bigcap b \cdot c \neq \emptyset$, $x/a \bigcap b \cdot c \neq \emptyset$, then $x \in a \cdot (b \cdot c)$, therefore $(a \cdot b) \cdot c \subset a \cdot (b \cdot c)$. The reverse inclusion can be proved similarly.
We may assert that $(S', \cdot)$ is a join space. Clearly e is an identity of $(S', \cdot)$.
Since $\{a, e\}$ is a closed subhypergroup and $\{a, e\} \subset \langle a \rangle$, it follows $\langle a \rangle = \{a, e\}$.
Thus $\langle a \rangle$ has the cardinal number equal to 2, if $a \neq e$.

Theorem 2.3. *Let $\langle J, \cdot \rangle$ be a join space with identity e in which, for any $a \in J \setminus \{e\}$, the closed subhypergroup $\langle a \rangle$ generated by a has the cardinal number equal to* 2. *Then $\langle J, \cdot \rangle$ is an associated join space of a projective geometry.*

Proof. Let $S = J \setminus \{e\}$ and let $L(ab) = \langle a, b \rangle \setminus \{e\}$, if $a, b \in S$, $a \neq b$. Each $L(ab)$ is a line and the set of all $L(ab)$ is denoted by T. One proves that (S, T) is a projective geometry, that is, it verifies the axioms (P1)-(P3) and finally that $\langle J, \cdot \rangle$ is an associated join space. For this, since $J = S \bigcup \{e\}$, we take e as the ideal point to be adjoined to S and $S' = J$.
If $J = \{e\}$, then $S = \emptyset$, $T = \emptyset$ and $\langle J, \cdot \rangle$ is an associated join space of (S, T), by definition.
If $J = \{a, e\}$, $a \neq e$, then $S = \{a\}$, $T = \emptyset$. We have $a \cdot e = e \cdot a = a$, $e \cdot e = e$, $e \in a \cdot a$. Thus $a \cdot a = e$ or $a \cdot a = \{a, e\}$ and, in the both cases, $\langle J, \cdot \rangle$ is an associated join space of (S, T), by definition.
Suppose the cardinal number of J is at least 3. Then S has at least two elements and $T \neq \emptyset$. Thus the join operation in S', denoted " $\circ$ ", is defined by:

(i) if $a, b \in S$, $\quad a \neq b$, then $a \circ b = L(ab) \setminus \{a, b\}$;

(ii) let $a \in S$; if some line of T contains exactly three points, then $a \circ a = e$; otherwise $a \circ a = \{a, e\}$;

(ii) if $a \in S'$, $e \circ a = a \circ e = a$.

To complete the proof, one shows that the hyperoperations " $\cdot$ " and " $\circ$ " are identical.

The previous two theorems characterize projective geometries in terms of join spaces in this way:
A join space $\langle J, \cdot \rangle$ is an associated join space of a projective geometry if and only if it has an identity e and for any $a \in J \setminus \{e\}$, the closed subhypergroup $\langle a \rangle$ generated by a has the cardinal number equal to 2.

2. The spherical geometry is the geometry of the two-dimensional surface of a sphere. It is an example of a non-Euclidian geometry, the simplest model for an elliptic geometry, where, given a line L and a point p outside L, there exists no line parallel to L passing through p.

Definition 2.4. A *spherical geometry* is a system $(S, (abc))$ involving a set S of points and a ternary *betweenness* relation (abc), which may be read *b lies between a and c*, which satisfies the following postulates.

(S1) If (abc) then a, b, c are distinct.

(S2) If (abc) then (cba).

(S3) For each a there exists a unique a' (called the *opposite* of a) such that $a' \neq a$ and (axy) always implies (xya').

(S4) If $b \neq a, \quad b \neq a'$, there exists x such that (axb).

If $b \neq a, b \neq a'$, we define $a \cdot b = \{x \mid (axb)\}$, $a \cdot a = a$. The hyperoperation " $\cdot$ " is a partial join operation in S; it is extended to subsets A, B of S in the familiar way $A \cdot B = \bigcup_{\substack{a \in A \\ b \in B}} a \cdot b$.

(S5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ provided both terms are defined.

Example. If we take S to be a Euclidean n-sphere and (abc) to mean b is an interior point of the minor arc of a great circle which joins a and c, then the axioms (S1)-(S5) are satisfied.

It remains the problem how to extend the definition of join to pairs of opposite points a, a' in such a way as to conserve the associative law. W.Prenowitz has proved that it is impossible to do this except in trivial cases (see [46]). Thus, with trivial exceptions, it is impossible to convert a spherical geometry $(S, (abc))$ into a join space $(S, \cdot)$. However, the conversion can easily be made by adjoining to S an "ideal point" which will play the role of an identity.

Let $S' = S \bigcup \{e\}$, $e \notin S$. The hyperoperation " $\cdot$ " is extended to S' as follows $a \cdot a' = \{a, a', e\}, \forall\, a \in S$, $a \cdot e = e \cdot a = a, \quad \forall\, a \in S'$.
Thus, in S' take place the commutative and the associative laws.

Theorem 2.5. *If S is a spherical geometry then S' is a join space with the identity element e, satisfies the idempotent law $aa = a$ and has the property that the closed subhypergroup $\langle a \rangle$ generated by $a \neq e$ has the cardinal number equal to* 3.
S' is called the associated join space of the spherical geometry S.

Proof. The axiom (S4) implies the axiom (JS1): $a \cdot b \neq \emptyset$. The axioms (JS2) and (JS3) are evident. We show $(S', \cdot)$ is a join space by verifying (JS4) and (JS5). Suppose $a, b, c, d \in S'$ such that $a/b \bigcap c/d \neq \emptyset$.
Using that $a/b = a \cdot b'$, we have $p \in a/b = a \cdot b'$, $p \in c/d = c \cdot d'$ and thus $a \in p/b' = p \cdot b = b \cdot p \in b \cdot (c \cdot d') = (b \cdot c) \cdot d'$. Then $a/d' \bigcap b \cdot c \neq \emptyset$, so that $a \cdot d \bigcap b \cdot c \neq \emptyset$.
(JS5) is immediate from $a/b = a \cdot b'$. Then one proves that $\langle a \rangle = \{a, a', e\}$. Clearly, $a/a = \{a, a', e\}$ is a closed subhypergroup contained by $\langle a \rangle$ and it is the least closed subhypergroup which contains a, therefore $\langle a \rangle = \{a, a', e\}$.

Theorem 2.6. *Let G be an idempotent join space with the identity e (that is, an idempotent regular hypergroup) with the property that for each $a \in G \setminus \{e\}$, the* *closed subhypergroup $\langle a\rangle$generated by a has the cardinal number equal to* 3*. Then* G *is the join space associated to a spherical geometry.*

Proof. Let $S = G \setminus \{e\}$ and we define (abc) to mean $b \in a \cdot c$, where $c \notin \{a, a'\}$ (in a regular hypergroup each element has an opposite). We show that S, with the order so defined, is a spherical geometry and that its associated join space S' coincides with G. First we show for $a \neq e$, that $a \cdot a' = \{a, a', e\}$.
Adding a to the both members of the relation $e \in a \cdot a'$ we have
$a = a \cdot e \in a \cdot (a \cdot a') = (a \cdot a) \cdot a' = a \cdot a'$ and similarly
$a' = e \cdot a' \in (a \cdot a') \cdot a' = a \cdot (a' \cdot a') = a \cdot a'$. Since $card\langle a\rangle = 3$, it results $a \cdot a' = \{a, a', e\}$.
To prove (S1), let us consider (abc), so that $b \in a \cdot c, \quad c \neq a, \quad c \neq a'$. Suppose $a = b$ then $a \in a \cdot c = c \cdot a$ and $c \in a/a \subset \langle a\rangle = \{a, a', e\}$ false. Similarly for $b = c$; so a, b, c are distinct.
(S2) follows from the commutative law of a join space.
(S3) can be verified by taking $p = a' = e/a$.
(S4) follows from $a \cdot b \neq \emptyset$, so that there exists $x \in S'$ such that $x \in a \cdot b$. Thus (axb).
To verify (S5) consider the operation " $\odot$ " defined in S by $a \odot a = a$ and $a \odot b = \{x \mid (axb)\}$ for $b \neq a, \quad b \neq a'$. We see immediately that $a \odot b = a \cdot b$, for $a, b \in S$ provided $b \neq a'$. Thus, since " $\cdot$ " is associative, the associative law for " $\odot$ " certainly holds for those triples $a, b, c \in S$ for which it is significant. Hence (S5) is verified and S is a spherical geometry.
Now we prove that $S' = S \bigcup \{e\}$ is the associated join space of S.
We extend " $\odot$ " to S' as it follows:
$a \odot a' = \{a, a', e\}, \forall\, a \in S, \quad a \odot e = e \odot a = a, \forall\, a \in S'$. As we have already proved, $a \cdot b = a \odot b, \forall\, a, b \in S$ and therefore $S' = G$.

3. The descriptive geometry, introduced by Gaspard Monge in 1975, is a branch of the geometry which deals with the study of the two-dimensional representation of the three-dimensional objects.

Definition 2.7. A *descriptive geometry* is a system $(S, (abc))$, where S is a set of elements called *points* and (abc) a ternary relation in S called *betweenness*, which satisfies the following postulates:

(D1) If a, b, c are points and (abc) then a, b, c are distinct.

(D2) If a, b, c are points and (abc) then (bca) is false.

If $a, b \ (a \neq b)$ are points, the set consisting of a, b and all points x for which (xab) or (axb) or (abx) is called the *line* ab. The set of points x for which (axb) is called the *segment* ab. The set of points x satisfying (xab) is called a *ray* and it is said to *emanate* from a.

(D3) If $c, d \ (c \neq d)$ are points of the line ab then a is a point of the line cd.

(D4) If $a, b \ (a \neq b)$ are points there is at least one point c such that (abc).

(D5) There exist three points not in the same line.

(D6) (Transversal postulate) If a, b, c are distinct points and a is not in the line bc and if d, e are points such that (bcd) and (cea) then there is a point f in the line de such that (afb).

A join operation is defined in S as follows:
if $a \neq b$, $a \cdot b = \{x \mid (axb)\}$ and $a \cdot a = a$ and the extension of a from b is $a/b = \{x \mid a \in x \cdot b\}$ and clearly these operations satisfy the properties:

(i) $a \cdot b \neq \emptyset$.

(ii) $a \cdot b = b \cdot a$.

(iii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(iv) $a/b \neq \emptyset$.

(v) $a/a = a = a \cdot a$.

(vi) $a/b \bigcap c/d \neq \emptyset \Longrightarrow a \cdot d \bigcap b \cdot c \neq \emptyset$.

The last property is a formulation of a triangle postulate employed by Peano: *Segments which join two vertices of a triangle to respective point of their opposite sides intersect.*

Thus $(S, \cdot)$ is a join space called *the associated join space* of the descriptive geometry $(S, (abc))$.

Notice that the line ab is the set $a \cdot b \bigcup a/b \bigcup b/a \bigcup \{a, b\}$.

Let us consider a join space $\langle J, \cdot \rangle$ for which $\forall\, a \in J,\quad a \cdot a = a/a = \{a\}$.
Define on J the following ternary relation: (abc) means $a \neq c$ and $b \in a \cdot c$.

Proposition 2.8.

(i) *If* (abc) *then* $a \cdot b \bigcap b \cdot c = \emptyset$.

(ii) *If* (abc) *then* a, b, c *are distinct.*

(iii) *If* (abc) *and* (bcd) *then* (abd) *and* (acd).

(iv) *If* (abc) *and* (acd) *then* (abd) *and* (bcd).

(v) (abx) *and* (aby) *imply the falsity of* (xay) *and* (xby). *Similarly,* (axb) *and* (ayb) *imply the falsity of* (xay) *and* (xby).

The following theorem establishes a first connection between the conditions of a join space and the postulates of a descriptive geometry.

Theorem 2.9. *The ternary relation in J satisfies the postulates* (D1), (D2), (D4), (D6).

Proof. (D1) is the assertion (ii) of the previous proposition.
By (iii), (abc) and (bca) imply (aba), a contradiction with (ii). Thus (D2) is valid.
(D4) is a restatement of the fact that in a join space $a/b \neq \emptyset$.
We verify (D6). Suppose a, b, c distinct, a does not belong to the line bc and $(bcd), (cea)$. Then $c \in b \cdot d$ and $e \in c \cdot a$. We obtain $a \cdot b \bigcap e/d \neq \emptyset$.
Let $f \in a \cdot b \bigcap e/d \neq \emptyset$. Then (afb).
If $d \neq e$, then f belongs to the line de.
If $d = e$ then (bcd) and (cea) imply (bca), so that a belongs to the line bc, contrary with the hypothesis. Thus $d \neq e$ and the proof is complete.

Remark 2.10. Since the direct sum of two join spaces is a join space, W.Prenowitz has shown that the postulate (D3) is independent of the conditions of a join space definition.

In the following, let $\langle J, \cdot \rangle$ be a join space, satisfying

(J0) $\forall\, a \in J, a \cdot a = a/a = a$.

We introduce three new postulates for J in order to characterize a descriptive geometry.

(J1) If $(a, b) \in J^2$, $\ a \neq b$, then $\langle a, b \rangle$ covers a.

This is a consequence of the postulate "two points belong to a unique line".

Remark 2.11.

(i) A join space satisfying (J1) is an exchange space.

(ii) (J1) is independent of conditions of join space definition and of condition (J0).

(J2) There exist A and B, closed hypergroups of $(J, \cdot)$ such that $B \subset A$ and the factor space $(A : B)$ has order 3.

It means that J contains two closed hypergroups A, B such that B separates A. (J2) is verified in a descriptive geometry, since we can take A to be a line and B one of its points.

The postulate
(J3) $d(J) > 2$
means that J contains a set of three independent elements.

Proposition 2.12. *For any* $(a, b) \in J^2$ *we have*

$$\langle a, b \rangle = a \cdot b \cup a/b \cup b/a \cup \{a, b\}.$$

Theorem 2.13. *Descriptive geometries are characterized as join spaces satisfying (J0), (J1), (J2), (J3).*

Proof. Let $(S, (abc)$ be a descriptive geometry. For any $(x, y) \in S^2$ we define $x \cdot x = x$ and $x \cdot y = \{t \mid (xty)\}$. Then $\langle S, \cdot \rangle$ is a join space, which verifies (J0), (J1), (J2), (J3), as we have seen before.

Conversely, if $\langle J, \cdot \rangle$ is a join space satisfying (J0), then (D1), (D2), (D4), (D6) hold, by the Theorem 2.9.
It remains to show (J1), (J2), (J3) imply (D3) and (D5).

The line ab (where $a \neq b$) is the set $a \cdot b \bigcup a/b \bigcup b/a \bigcup \{a, b\}$ which coincides with $\langle a, b \rangle$. We verify that if $a \neq b$ and $\{c, d\} \subset \langle a, b \rangle$,where $c \neq d$, then $\langle a, b \rangle = \langle c, d \rangle$ and this obviously implies (D3). Suppose $c \neq a$, then $\langle a, b \rangle \supset \langle c, d \rangle \ni a$ and by (J1) it follows $\langle a, b \rangle = \langle a, c \rangle$. Thus $d \in \langle a, c \rangle$. Since $c \neq d$, we can similarly show $\langle a, c \rangle = \langle c, d \rangle$ and then $\langle a, b \rangle = \langle c, d \rangle$.

By (J3) there are three distinct elements a, b, c of J, which form an independent set. Suppose a, b, c are contained in the same line $pq = \langle p, q \rangle$. But then $\langle a, b \rangle = \langle p, q \rangle \ni c$, contrary to the independence of a, b, c. Therefore a, b, c are not in the same line and (D5) is verified.

Chapter 3

Hypergroups and fuzzy sets

In the last decades, many connections between hypergroups and fuzzy sets have been established in order to obtain new algebraic structures, connections studied from a theoretical point of view, but also for their applications obtained in various domains: the graphs theory, the probabilistic spaces theory, for example.

The "fuzzy" concepts derive from the various fuzzy phenomena presented in the nature and in our life. For example, the rain is a commune natural phenomenon, which can not be described with precision, since it can rain with varying intensity anywhere from a light sprinkle to a torrential downpour. But the phenomenon can be characterized by a function which associates a real number from the closed interval $[0, 1]$ to any kind of rain: if it doesn't rain the function takes the value 0 and the greater is the intensity of the rain, the nearer to 1 is the value of the function.

The concept of fuzzy set has been introduced by L.A.Zadeh in 1965 (see [58]), when he proposed the idea of a multi-valued logic, which extends the traditional concept of a bivalent logic, which becomes a particular case of the new theory. The fuzzy set theory is based on the principle called by L.A.Zadeh "the principle of incompatibility", that is "the closer a phenomenon is studied, the more indistinct its definition becomes".

In general, a fuzzy set of an arbitrary universe X is defined by a function $\mu_A : X \longrightarrow [0, 1]$, where A is a non-empty subset of X. For any element $x \in X$, the value $\mu_A(x)$ is called the membership degree of x to A. When A is a set in the classical sense, then $\mu_A(x) = 0$, if $x \notin A$ and $\mu_A(x) = 1$, if $x \in A$, that is, μ_A is the characteristic function of the set A.

3.1 The fuzzy subgroup of a group

Since has been introduced the concept of fuzzy set, some notions have been extended and generalized, substituting the classical sets with the fuzzy sets. The first connection between fuzzy sets and algebraic structures has been considered by A.Rosenfeld in 1971 (see [48]), when he defined the notion of fuzzy subgroup of a group $(G, \cdot)$, with applications in the economical mathematics (see [3]). Later, R.Ameri and M.M.Zahedi have extended this association for constructing a new hyperstructure $\langle G, \circ_\mu \rangle$, which under suitable conditions is a hypergroup, or moreover, a join space (see [1]).

In the following, we present some results obtained in these directions.

Definition 3.1.1. Let $(G, \cdot)$ be a group and μ be a fuzzy set of G. Then μ is called *fuzzy subgroup* of G if

(i) $\forall\, x, y \in G, \mu(xy) \geq \min(\mu(x), \mu(y))$;

(ii) $\mu(x^{-1}) \geq \mu(x), \forall\, x \in G$.

For a non-empty set X, we denote

$$FS(X) = \{\text{all non zero function } \ \mu{:}X \longrightarrow [0,1]\},$$

named the set of all (non zero) fuzzy subsets of X. From now by G we mean a group with the identity e.

Proposition 3.1.2. *We have that μ is a fuzzy subgroup of G if and only if*

$$\mu(xy^{-1}) \geq \min(\mu(x), \mu(y)), \forall\, x, y \in G.$$

Proof. If μ is a fuzzy subgroup we have $\mu(x) = \mu((x^{-1})^{-1}) \leq \mu(x^{-1}) \leq \mu(x)$, so $\mu(x^{-1}) = \mu(x)$. Then $\mu(xy^{-1}) \geq \min(\mu(x), \mu(y^{-1})) = \min(\mu(x), \mu(y))$.

Conversely, if $\mu(xy^{-1}) \geq \min(\mu(x), \mu(y)), \forall\, x, y \in G$, let $y = x$ to obtain $\mu(e) \geq \mu(x), \forall\, x \in G$; hence $\mu(y^{-1}) = \mu(ey^{-1}) \geq \min(\mu(e), \mu(y)) = \mu(y)$ and $\mu(xy) = \mu(x(y^{-1})^{-1}) \geq \min(\mu(x), \mu(y^{-1})) \geq \min(\mu(x), \mu(y))$.

Definition 3.1.3. $\mu \in FS(G)$ is called

(i) *symmetric* if $\mu(x) = \mu(x^{-1}), \forall x \in G$;

(ii) *invariant* if $\mu(xy) = \mu(yx), \forall x, y \in G$;

(iii) *subnormal* if it is both symmetric and invariant.

Definition 3.1.4. Let $\mu \in FS(G)$ and $x \in G$. Then *the left fuzzy coset* of μ, denoted $x\mu \in FS(G)$, is defined by $(x\mu)(g) = \mu(x^{-1}g)$, for any $g \in G$.
Similarly, *the right fuzzy coset* of μ, denoted $\mu x \in FS(G)$, is defined by $(\mu x)(g) = \mu(gx^{-1})$, for any $g \in G$.

Notations. Let $\mu \in FS(G)$ and $a, b \in G$. Denote

${}^a\mu = \{x \in G \mid x\mu = a\mu\}$

$\mu^a = \{x \in G \mid \mu x = \mu a\}$

$a\mu^e = \{ax \mid x \in \mu^e\}$

$\mu^a\mu^b = \{xy \mid x \in \mu^a, y \in \mu^b\}$.

If μ is invariant then it is easy to see that ${}^a\mu = \mu^a, \forall a \in G$.

Lemma 3.1.5. *Let $\mu \in FS(G)$ be subnormal. Then*

(i) $\forall x, y \in G, x\mu = y\mu \iff y^{-1}x \in \mu^e$;

(ii) μ^e *is a normal subgroup of* G;

(iii) $\mu^a = a\mu^e, \forall a \in G$;

(iv) $\mu^a\mu^b = \mu^{ab}, \forall a, b \in G$;

(v) $x\mu = y\mu \iff zx\mu = zy\mu, \forall x, y, z \in G$;

(vi) $x\mu = y\mu \iff xz\mu = yz\mu, \forall x, y, z \in G$.

Proof. (i) Let $x, y \in G$ such that $x\mu = y\mu \implies \forall g \in G, \mu(x^{-1}g) = \mu(y^{-1}g)$. For $g \mapsto xg$ we have $\mu(g) = \mu(y^{-1}xg) \implies y^{-1}x \in \mu^e$.
Similarly the other implication.

(ii) We know $\mu^e \leq G \iff \forall x, y \in \mu^e \implies xy^{-1} \in \mu^e$.
Let $x, y \in \mu^e \implies \mu x = \mu y = \mu e$ and by (i) μ^e is a subgroup of G.
Moreover, $\mu^e \trianglelefteq G \iff \forall x \in G, \forall h \in \mu^e \implies xhx^{-1} \in \mu^e$.
Let $x \in G$ and $h \in \mu^e \implies \mu h = \mu e \implies \forall g \in G, \mu(h^{-1}g) = \mu(g)$.
It is obvious that $\forall g \in G, \mu(xh^{-1}x^{-1}g) = \mu(g)$.

(iii) Let $a \in G$ and $x \in \mu^a \implies \mu x = \mu a \implies a^{-1}x \in \mu^e \implies a^{-1}x = y$ with $y \in \mu^e \implies x = ay, y \in \mu^e \implies x \in a\mu^e \implies \mu^a \subset a\mu^e$.
Similarly the other inclusion.

(iv)We show $\mu^a\mu^b \subset \mu^{ab}$.
Let $xy \in \mu^a\mu^b \implies x \in \mu^a = a\mu^e, x = ax', x' \in \mu^e$ and $y \in \mu^b = b\mu^e, y = by', y' \in \mu^e$. Since μ^e is a normal subgroup of G, $a\mu^e = \mu^e a, \forall a \in G$. So $xy = ax'by' = abx'y' \in ab\mu^e = \mu^{ab}$.
Similarly the other inclusion.

(v) If $x\mu = y\mu$ then $xy^{-1} \in \mu^e$; we know $\mu^e \trianglelefteq G$ and thus $zx\mu = zy\mu \iff zx(zy)^{-1} = zxy^{-1}z^{-1} \in \mu^e$.
We suppose now $zx\mu = zy\mu, \forall z \in G$. For $z = e$ we have $x\mu = y\mu$.

(vi)$x\mu = y\mu \iff xy^{-1} \in \mu^e \iff xzz^{-1}y^{-1} = xz(yz)^{-1} \in \mu^e \iff xz\mu = yz\mu$.

Let $\mu \in FS(G)$. We define the hyperoperation

$$\circ_\mu : G \times G \longrightarrow \mathcal{P}^*(G), a \circ_\mu\ b = \mu^a\mu^b$$

named the *hyperoperation induced* by μ.

The next result determines the conditions under which the hypergroupoid $\langle G, \circ_\mu \rangle$ is a hypergroup, a canonical hypergroup, a join space.

Theorem 3.1.6. *Let* $\mu \in FS(G)$.

(i) *Then* $\langle G, \circ_\mu \rangle$ *is a cuasihypergroup.*

(ii) *If* $\mu \in FS(G)$ *is subnormal, then* $\langle G, \circ_\mu \rangle$ *is a hypergroup.*

(iii) *If* $\mu \in FS(G)$ *is subnormal, then* $\langle G, \circ_\mu \rangle$ *is a polygroup. Moreover, there exists a good homomorphism between* $\langle G, \cdot \rangle$ *and* $\langle G, \circ_\mu \rangle$.

(iv) *The hypergroup* $\langle G, \circ_\mu\rangle$ *is a join space if and only if the commutator subgroup* $[G, G] \subseteq \mu^e$*, where* e *is the identity of the group* G.

Proof. (i) We have to prove the reproductive law: $a \circ_\mu G = G = G \circ_\mu a, \forall a \in G$. This is clear because for $x \in G$, $x \in a \circ_\mu (a^{-1}x) = \mu^a \mu^{a^{-1}x} = \mu^x$ and $x \in (xa^{-1}) \circ_\mu a = \mu^{xa^{-1}} \mu^a = \mu^x$.

(ii) It is enough to show that " $\circ_\mu$ " is associative. Let $a, b, c \in G$. Then

$$(a \circ_\mu b) \circ_\mu c = \bigcup_{x \in \mu^a \mu^b} \mu^x \mu^c = \bigcup_{x \in \mu^{ab}} \mu^{xc},$$

since μ is subnormal.
But $x \in \mu^{ab}$ implies $\mu x = \mu ab$ and therefore $(a \circ_\mu b) \circ_\mu c = \mu^{(ab)c}$.
Similarly, $a \circ_\mu (b \circ_\mu c) = \mu^{a(bc)}$ and by consequence $(a \circ_\mu b) \circ_\mu c = a \circ_\mu (b \circ_\mu c)$.

(iii) Let $a \in G$. Then $a \in \mu^a = \mu^{ea} = \mu^e \mu^a = e \circ_\mu a$ and $a \in \mu^a = \mu^{ae} = \mu^a \mu^e = a \circ_\mu e$. So $a \in e \circ_\mu a \cap a \circ_\mu e$. Similarly, $e \in a \circ_\mu a^{-1} \cap a^{-1} \circ_\mu a$. Now suppose $c \in a \circ_\mu b = \mu^{ab}$. Thus $\mu c = \mu ab$ and so $\mu a = \mu c b^{-1}$, that is $a \in c \circ_\mu b^{-1}$. Also, $b \in a^{-1} \circ_\mu c$. Therefore $(G, \circ_\mu)$ is a polygroup.

Now we define $\varphi : (G, \cdot) \longrightarrow (G, \circ_\mu)$, $\varphi(a) = \mu^a$. Then $\varphi(ab) = \mu^{ab} = \mu^a \mu^b$ and $\varphi(a) \circ_\mu \varphi(b) = \mu^a \circ_\mu \mu^b = \bigcup_{x \in \mu^a, y \in \mu^b} x \circ_\mu y = \bigcup_{xy \in \mu^{ab}} \mu^{xy} = \mu^{ab} = \varphi(ab)$.
Thus φ is a good homomorphism.

(iv) Now we check the commutativity and the transposition properties of $\langle G, \circ_\mu\rangle$. Let $a, b \in G$. Then
$a \circ_\mu b = b \circ_\mu a \Longleftrightarrow \mu^{ab} = \mu^{ba} \Longleftrightarrow ab\mu = ba\mu \Longleftrightarrow a^{-1}b^{-1}ab \in \mu^e \Longleftrightarrow [G, G] \subseteq \mu^e$.
Thus $\langle G, \circ_\mu\rangle$ is a commutative hypergroup if and only if $[G, G] \subseteq \mu^e$.

Let $a, b, c, d \in G$ and $a/b \cap c/d \neq \emptyset$. Then $\exists x \in a/b \cap c/d$ such that $a \in x \circ_\mu b$ and $c \in x \circ_\mu d$. Thus $\mu a = \mu x b$ and $\mu c = \mu x d$. But

1) $\mu a = \mu x b \Longrightarrow \mu a d = \mu x b d$

2) $\mu c = \mu x d \Longrightarrow \mu b c = \mu b x d$.

From 1) and 2) we have
$\mu a d = \mu b c \Longleftrightarrow (xbd)(bxd)^{-1} \in \mu^e \Longleftrightarrow xbx^{-1}b^{-1} \in \mu^e \Longleftrightarrow [G, G] \subseteq \mu^e$.
Therefore $\langle G, \circ_\mu\rangle$ is a join space $\Longleftrightarrow [G, G] \subseteq \mu^e$.

3.2 The classic association between fuzzy sets and hypergroups

In [9], Corsini established a new connection between hypergroups and fuzzy sets; afterwards, P.Corsini and V.Leoreanu have obtained more results concerning this association (see [11], [13], [14]); some of them are presented in the following.

Let $\mu : H \longrightarrow I$ be a function from a non-empty set H to the closed interval $I = [0, 1]$, that is $\langle H; \mu \rangle$ is a fuzzy subset. Let us define on H the hyperoperation: for all $(x, y) \in H^2$,

$$x \circ y = y \circ x = \{z \in H \mid \min\{\mu(x), \mu(y)\} \leq \mu(z) \leq \max\{\mu(x), \mu(y)\}\}.$$

Theorem 3.2.1. *The hypergroupoid* $\langle H, \circ \rangle$ *is a join-space.*

Proof. It is clear that the hyperoperation "$\circ$" is associative and reproducible, that is $\langle H, \circ \rangle$ is a commutative hypergroup. It remains to prove that $\langle H, \circ \rangle$ satisfies the condition

$$\text{for all } \ (a, b, c, d) \in H^4, a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

Let us suppose $x \in a/b \cap c/d$; then $\mu(a) \in [\mu(x), \mu(b)]$ and $\mu(c) \in [\mu(x), \mu(d)]$. We distinguish four cases:

1) $\mu(x) \leq \mu(b), \mu(x) \geq \mu(d)$.
   Then $\mu(x) \leq \mu(a) \leq \mu(b), \mu(d) \leq \mu(c) \leq \mu(x)$, from which it results $\mu(d) \leq \mu(c) \leq \mu(x) \leq \mu(a) \leq \mu(b)$, whence $[\mu(c), \mu(x)] \subset [\mu(d), \mu(a)] \cap [\mu(c), \mu(b)]$ and thus $a \circ d \cap b \circ c \neq \emptyset$.

2) $\mu(x) \geq \mu(b), \mu(x) \leq \mu(d)$.
   It follows $\mu(b) \leq \mu(a) \leq \mu(x)$ and $\mu(x) \leq \mu(c) \leq \mu(d)$; then $\mu(b) \leq \mu(a) \leq \mu(x) \leq \mu(c) \leq \mu(d)$, which implies $[\mu(a), \mu(x)] \subset [\mu(b), \mu(c)] \cap [\mu(a), \mu(d)]$, therefore $a \circ d \cap b \circ c \neq \emptyset$.

3) $\mu(x) \leq \mu(b), \mu(x) \leq \mu(d)$.
   Then $\mu(a) \leq \mu(b)$ and $\mu(c) \leq \mu(d)$. We consider

   (i) $\mu(b) \leq \mu(d)$ and then $\mu(a) \leq \mu(b) \leq \mu(d)$, thus $a \circ d \cap b \circ c \neq \emptyset$.

   (ii) $\mu(b) \geq \mu(d)$ and therefore $\mu(c) \leq \mu(d) \leq \mu(b)$, so again $b \circ c \cap a \circ d \neq \emptyset$.

4) $\mu(x) \geq \mu(b), \mu(x) \geq \mu(d)$.
   In this case we have $\mu(b) \leq \mu(a)$ and $\mu(d) \leq \mu(c)$. There are two possibilities:

   (i) $\mu(a) \leq \mu(c)$, with $\mu(b) \leq \mu(a) \leq \mu(c)$

   (ii) $\mu(a) \geq \mu(c)$, with $\mu(d) \leq \mu(c) \leq \mu(a)$

   In the both cases $a \circ d \cap \ b \circ c \neq \emptyset$.

Theorem 3.2.2.

1) *We have, for all* $n \in \mathbb{N}^*$, *for all* $(z_1, z_2, ..., z_n) \in H^n$,

$$\prod_{i=1}^{n} z_i = \left\{ u \in H \mid \bigwedge_{i=1}^{n} \mu(z_i) \leq \mu(u) \leq \bigvee_{i=1}^{n} \mu(z_i) \right\}.$$

   Moreover, if $\mathcal{R}$ *is the equivalence relation defined on* H*:*

$$x\mathcal{R}y \Longleftrightarrow \mu(x) = \mu(y), \quad \textit{then} \quad \mathcal{R} \subset \beta_2;$$

2) *$H/\mathcal{R}$ is a hypergroup with respect to the hyperoperation*

$$\bar{x} \otimes \bar{y} = \{\bar{z} \mid z \in x \circ y\}.$$

Proof. 1) It follows inductively from the definition. Indeed we have

$$\prod_{i=1}^{n} z_i = \prod_{i=1}^{n-1} z_i \circ z_n = \bigcup_{v \in \prod_{i=1}^{n-1} z_i} v \circ z_n = \bigcup_{v \in \prod_{i=1}^{n-1} z_i} \{\lambda \mid \mu(v) \wedge \mu(z_n) \leq \mu(\lambda) \leq \mu(v) \vee \mu(z_n)\}.$$

By induction suppose

$$\prod_{i=1}^{n-1} z_i = \left\{ \delta \mid \bigwedge_{i=1}^{n-1} \mu(z_i) \leq \mu(\delta) \leq \bigvee_{i=1}^{n-1} \mu(z_i) \right\}.$$

Then we obtain

$$\prod_{i=1}^{n} z_i = \bigcup_{\bigwedge_{i=1}^{n-1} \mu(z_i) \leq \mu(v) \leq \bigvee_{i=1}^{n-1} \mu(z_i)} \{\mu(v) \wedge \mu(z_n) \leq \mu(\lambda) \leq \mu(v) \vee \mu(z_n)\} =$$

$$= \left\{ \lambda \mid \bigwedge_{i=1}^{n} \mu(z_i) \leq \mu(\lambda) \leq \bigvee_{i=1}^{n} \mu(z_i) \right\}.$$

Since $\langle H, \circ \rangle$ is a join space, it is a hypergroup, whence, for all $(a,b) \in H^2$, there is $q \in H$ such that $a \in b \circ q$. So, if $a\mathcal{R}a'$, it follows $a' \in \mu^{-1}\mu(a) \subset b \circ q$, then $\mathcal{R}(a) \subset b \circ q$ and therefore $\mathcal{R} \subset \beta_2$.

2) It is enough to prove that $\mathcal{R}$ is regular. Indeed, $a\mathcal{R}a'$, $b\mathcal{R}b'$ implies $a \circ b = a' \circ b'$. Then, by the Theorem 3.2.1, $\langle H/\mathcal{R}, \otimes \rangle$ is a hypergroup.

In the following, we shall give a necessary and sufficient condition for the isomorphism of two join spaces, associated with two different fuzzy subsets, on the same universe H.

First we study the case of a finite universe.

Let us consider $H = I(n) = \{1, 2, ..., n\}$ and μ_A a fuzzy subset on H. We define on H the equivalence relation

$$u \sim_A v \text{ if and only if } \mu_A(u) = \mu_A(v).$$

We denote $H' = H/\sim_A$, $H' = \{\bar{h}_1, \bar{h}_2, ..., \bar{h}_s\}$ and we order H' such that

I) $\forall\, (h, k) \in H^2, \overline{h} < \overline{k}$ if and only if $\mu_A(h) < \mu_A(k)$.

Let $\lambda(\mu_A)$ be the ordered partition of n into s parts defined as follows:

II) $\forall (\bar{h}_1, \bar{h}_2, ..., \bar{h}_s) \in H'^s, \lambda(\mu_A) = (a_1, a_2, ..., a_s)$ if and only if $\forall i$, $a_i = \mid \mu_A^{-1}(\mu_A(h_i)) \mid$ and $\forall (i,j) \in I(s) \times I(s)$ such that $i \neq j$, $i < j \Longrightarrow \bar{h}_i < \bar{h}_j$.

Clearly, we have $\sum_{i=1}^{s} a_i = n$ and $\forall i, a_i \geq 1$.
Let $(a_1, a_2, ..., a_s)$ be an ordered partition of n into s parts. Let us set

III) $\psi(a_1, a_2, ..., a_s) = (b_1, b_2, ..., b_s)$ where for each $i, 1 \leq i \leq s, b_i = a_{s-i+1}$.

Theorem 3.2.3. *If μ_A, μ_B are fuzzy subsets on a finite universe H, then the join spaces $\langle H; \circ_A \rangle$ and $\langle H; \circ_B \rangle$ are isomorphic if and only if either $\lambda(\mu_A) = \lambda(\mu_B)$ or $\lambda(\mu_B) = \psi(\lambda(\mu_A))$.*

Proof. First we prove the implication "$\Longleftarrow$".
Let us suppose $\lambda(\mu_A) = \lambda(\mu_B) = (a_1, a_2, ..., a_s)$. We can set
$H = \bigcup_{j=1}^{s} H_j = \bigcup_{j=1}^{s} H'_j$, where $\forall j \in I(s), H_j = \mu_A^{-1}(\mu_A(h_j))$ and $H'_j = \mu_B^{-1}(\mu_B(h_j))$.
We take $H_j = \{x_{1,j}, x_{2,j}, ..., x_{a_j,j}\}, H'_j = \{x'_{1,j}, x'_{2,j}, ..., x'_{a_j,j}\}$ and we order H in the following manner

$$\forall (j, j') \in I(s) \times I(s), \forall (h, h') \in I(a_j) \times I(a_j)$$

$$x_{h,j} < x_{h',j} \Longleftrightarrow h < h'.$$

If $j \neq j'$, for all $(h, h') \in I(a_j) \times I(a'_j)$,

$$x_{h,j} < x_{h',j'} \Longleftrightarrow j < j'.$$

Moreover, for all $(i,j) \in I(s) \times I(s)$ we denote $i \vee j = \max\{i,j\}, i \wedge j = \min\{i,j\}$.
Then, for all $(i,j) \in I(s) \times I(s)$ and for all $(h,k) \in I(a_i) \times I(a_j)$, we have

$$x_{h,i} \circ_A x_{k,j} = \bigcup_{i \wedge j \leq r \leq i \vee j} H_r,$$

$$x'_{h,i} \circ_B x'_{k,j} = \bigcup_{i \wedge j \leq r \leq i \vee j} H'_r.$$

Therefore, if $f : \langle H; \circ_A \rangle \longrightarrow \langle H; \circ_B \rangle$ is the function defined as follows: for each $(u,t) \in I(a_t) \times I(s), f(x_{u,t}) = x'_{u,t}$, then we have

$$f(x_{h,i} \circ_A x_{k,j}) = x'_{h,i} \circ_B x'_{k,j} = f(x_{h,i}) \circ_B f(x_{k,j}),$$

whence $\langle H; \circ_A \rangle$ and $\langle H; \circ_B \rangle$ are isomorphic hypergroups.
Let us suppose now $\lambda(\mu_B) = \psi(\lambda(\mu_A))$.
We consider $H = \bigcup_{1 \leq j \leq s} H_j$, where, for any $j \in I(s)$, we denote
$H_j = \{x_{1,j}, x_{2,j}, ..., x_{a_j,j}\} = \mu_A^{-1}(\mu_A(h_j))$ and similarly, $H = \bigcup_{1 \leq j' \leq s} H'_{j'}$, where

$j' = \psi(j) = s - j + 1$ and $H'_{j'} = \{x'_{1,j'}, x'_{2,j'}, ..., x'_{a'_{j'},j'}\} = \mu_B^{-1}(\mu_B(h_j))$, with $a'_{j'} = a_j$.

Let us define for each $(h,j) \in I(a_j) \times I(s), f(x_{h,j}) = x'_{h,j'}$. We have

$$\begin{aligned} f(x_{h,i} \circ_A x_{k,j}) &= f\left(\bigcup_{i\wedge j \le r \le i \vee j} H_r\right) &= \bigcup_{\psi(i\vee j)\le \psi(r) \le \psi(i \wedge j)} H'_{\psi(r)} \\ &= \bigcup_{i'\wedge j' \le r' \le i' \vee j'} H'_{r'} &= x'_{h,i'} \circ_B x'_{k,j'} = f(x_{h,i}) \circ_B f(x_{k,j}), \end{aligned}$$

whence f is an isomorphism.

Let us prove now the implication " $\Longrightarrow$ ".

Let $p : H \longrightarrow I(s)$ be the function defined as follows:
for any $x \in H, p(x) = j$, where j is that unique element of $I(s)$, such that it exists $k \in I(a_j)$ so that $x = x_{k,j} \in H_j$. Similarly, we define $p' : H \longrightarrow I(s')$, where $s' = \mid H/\sim_B \mid$.

Let $f : \langle H; \circ_A \rangle \longrightarrow \langle H; \circ_B \rangle$ be the isomorphism of these two join spaces. Then, for any $(x,y) \in H^2$ we have

$$f(x \circ_A y) = f\left(\bigcup_{p(x)\wedge p(y) \le j \le p(x) \vee p(y)} H_j\right) = \bigcup_{p(x)\wedge p(y) \le j \le p(x) \vee p(y)} f(H_j).$$

But, setting $\mu_B^{-1}\mu_B(x'_{k,r}) = H'_r$, we also have

$$f(x \circ_A y) = f(x) \circ_B f(y) = \bigcup_{p'(f(x))\wedge p'(f(y)) \le r \le p'(f(x)) \vee p'(f(y))} H'_r.$$

For any $(u,v) \in I(n) \times I(n)$, we shall denote by $I(u,v)$ the set

$$\{z \in I(n) \mid u \wedge v \le z \le u \vee v\}.$$

We remark now that

(1) if $r_1 \neq r_2$, then $H'_{r_1} \cap H'_{r_2} = \emptyset$;

(2) if $\{x,y\} \subset H_j$, then $x \circ_A y = H_j = x \circ_A x = y \circ_A y$, whence

$f(x) \circ_B f(y) = f(x \circ_A y) = f(x \circ_A x) = f(H_j) = f(x) \circ_B f(x).$

(3) by (1) and (2) there exists a unique $t = p'(f(x))$ such that

$f(H_j) = f(x) \circ_B f(y) = H'_t.$

We set $t = \phi(j)$, whence $f(H_j) = H'_{\phi(j)}$.
So we have a function $\phi : I(p(x), p(y)) \longrightarrow I(p'(f(x)), p'(f(y)))$ and clearly, we obtain that, for any $x \in H, \phi(p(x)) = p'(f(x))$.

We shall prove that ϕ is a bijection.

ϕ is injective. Indeed, if there are j_1, j_2 such that $j_1 \neq j_2$ and $\phi(j_1) = \phi(j_2)$, it follows $f(H_{j_1}) = f(H_{j_2})$, whence, for any $k \in I(a_{j_1})$, there exists $h \in I(a_{j_2})$ such that $f(x_{k,j_1}) = f(x_{h,j_2})$, which is absurd since $H_{j_1} \cap H_{j_2} = \emptyset$ and f is an injective function.

ϕ is also surjective. Indeed, since

$$f(x_{k,i} \circ_A x_{h,j}) = f(x_{k,i}) \circ_B f(x_{h,j}) = \bigcup_{r \in I(p'(f(x_{k,i})), p'(f(x_{h,j})))} H'_r$$

and $p'(f(x_{k,i})) = \phi(i), p'(f(x_{h,j})) = \phi(j)$, we have that, for any $r \in I(\phi(i), \phi(j))$, there exists $t \in I(i,j)$ such that $\phi(t) = r$.

Therefore, ϕ is a bijection from the interval $I(i,j)$ to the interval $I(\phi(i), \phi(j))$. On the other hand, as f is a one-to-one function, it follows that for any $y \in H$, there exists an unique pair $(k,j) \in I(a_j) \times I(s)$ such that $f(x_{k,j}) = y$, whence

$$f(H) = H = \bigcup_{j \in I(s)} f(H_j) = \bigcup_{j \in I(s)} H'_{\phi(j)} = \bigcup_{r \in I(\phi(1), \phi(s))} H'_r.$$

Moreover, ϕ is a function from $I(s)$ to $I(s')$ and since $s' = \mid H/\sim_B \mid$ and f is an isomorphism, we have

$$\mid f(H)/\sim_B \mid = \mid H/\sim_B \mid = \mid p'(H) \mid = \mid p'(f(H)) \mid = \mid \phi(p(H)) \mid,$$

whence $s' = \mid H/\sim_B \mid = \mid \phi(p(H)) \mid = \mid p(H) \mid = s$.

Moreover, for any $j \in I(s)$, we have

$$a_j = \mid H_j \mid = \mid f(H_j) \mid = \mid H_{\phi(j)} \mid = a'_{\phi(j)}.$$

On the other hand, for any $k \in I(a_1)$ and for any $h \in I(a_s)$,

$$f(H) = f(x_{k,1} \circ_A x_{h,s}) = \bigcup_{t \in I(1,s)} H'_t = H.$$

Therefore, the interval $I(\phi(1), \phi(s))$ coincides with the interval $I(1,s) = I(s)$. It follows $\{\phi(1), \phi(s)\} = \{1, s\}$. Hence

$$\begin{aligned} I(2, s-1) &= I(1,s) \setminus \{1,s\} = I(\phi(1), \phi(s)) \setminus \{\phi(1), \phi(s)\} = \phi(I(s)) \setminus \{\phi(1), \phi(s)\} = \\ &= \phi(I((2, s-1)) = I(\phi(2), \phi(s-1)). \end{aligned}$$

From $I(2, s-1) = I(\phi(2), \phi(s-1))$, one obtains $\{\phi(2), \phi(s-1)\} = \{2, s-1\}$. In general, we have

(ε) for any k, $\phi(k) \in \{k, s-k+1\}$.

Let V be the set of the permutations of $I(s)$ which satisfy (ε).

(η) We shall prove now that either ϕ is the identity function I of $I(s)$ or it is the permutation Ψ of $I(s)$ defined:

$$\text{for any } \; k \in I(s), \Psi(k) = s - k + 1.$$

If $s \leq 3$ we have $V = \{I_{I(s)}, \Psi\}$. If $s > 3$ and one supposes $\phi(1) = 1, \phi(2) =$ $= \Psi(2) = s - 2 + 1 = s - 1$, it follows $\phi(I(1,2)) = I(\phi(1), \phi(2)) = I(1, s-1)$, whence $2 =\mid I(1,2) \mid=\mid \phi(I(1,2)) \mid\neq\mid I(1, s-1) \mid\geq 3$, absurd.

Analogously, if $\phi(1) = s$, then $\phi(2) = 2$ and we obtain a contradiction.

Therefore either $\phi_{I(2)} = I_{I(2)}$ or $\phi_{I(2)} = \Psi_{I(2)}$.

Let k be in $I(s)$ and let us suppose

$$\phi_{I(k)} = I_{I(k)}, \phi(k+1) = \Psi(k+1).$$

Then we have

$$k+1 =\mid I(k+1) \mid=\mid \phi(I(k+1)) \mid=\mid I(\phi(1), \phi(k)) \mid + \mid I(\phi(k), \Psi(k+1) \mid -1 =$$
$$= k + \mid I(k, s-k) \mid -1 = k + s - 2k, \quad \text{from which} \quad s = 2k+1,$$

hence $\phi(k+1) = s - (k+1) + 1 = k+1$ and then $\phi_{I(k+1)} = I_{I(k+1)}$.

Similarly, if one supposes

$$\phi_{I(k)} = \Psi_{I(k)}, \phi(k+1) = k+1$$

we have $k+1 =\mid \phi(I(k+1)) \mid=\mid \phi(I(k)) \mid + \mid \phi(I(k, s-k)) \mid -1 = k + s - 2k$, from which $s = 2k+1$. Then $\Psi(k+1) = s - (k+1) + 1 = k+1 = I(k+1)$ hence, $\phi_{I(k+1)} = \Psi_{I(k+1)}$.

Therefore, by induction, (η) is proved and consequently the implication " $\Longrightarrow$ " follows and the theorem is completed.

Now, it is interesting to study how two isomorphism problem of two join spaces associated with different fuzzy subsets on a finite universe can be generalized for the case of an arbitrary universe.

For this, we use the same notations: if μ_A and μ_B are two different fuzzy subsets on H, then $H/\sim_A = \{H_i \mid i \in I\}$ and $H/\sim_B = \{H'_{i'} \mid i' \in I'\}$, where

$$\begin{array}{llll} \forall\, i \in I, & H_i = \{x_{k,i} \mid k \in K_i\}; & \mid K_i \mid=\mid H_i \mid= a_i & \text{and} \\ \forall\, i' \in I', & H'_{i'} = \{x'_{k',i'} \mid k' \in K'_{i'}\}; & \mid K'_{i'} \mid=\mid H'_{i'} \mid= a'_{i'}. & \end{array}$$

We order I (I', respectively) such that:
$i < j \Longleftrightarrow$ for any $(x, y) \in H_i \times H_j, \mu_A(x) < \mu_A(y)$
($i' < j' \Longleftrightarrow$ for any $(x', y') \in H'_{i'} \times H'_{j'}, \mu_B(x') < \mu_B(y')$, respectively)
and we order also the quotient set $H/\sim_A$ ($H/\sim_B$, respectively) such that:
$H_i < H_j \Longleftrightarrow i < j$ ($H'_{i'} < H'_{j'} \Longleftrightarrow i' < j'$, respectively).

We obtain the following result.

Theorem 3.2.4. *If μ_A, μ_B are fuzzy subsets on a universe H, then the join spaces $\langle H; \circ_A \rangle$ and $\langle H; \circ_B \rangle$ are isomorphic if and only if a strictly monotone and bijective function $\varphi : I \longrightarrow I'$ exists, such that, for any $i \in I, a_i = a'_{\varphi(i)}$.*

Proof. First, let us prove the implication " $\Longrightarrow$ ".
We denote by f the isomorphism between $\langle H; \circ_A \rangle$ and $\langle H; \circ_B \rangle$.
As in the finite case, we shall consider the function $p : H \longrightarrow I, p(x) = j$, where j is the unique element of I such that $x \in H_j$ and, similarly, $p' : H \longrightarrow I'$, $p'(x) = j'$, where j' is the unique element of I' such that $x \in H'_{j'}$.

For $\{x,y\} \subset H_i$, we have $x \circ_A x = x \circ_A y = y \circ_A y = H_i$, so we obtain $f(x) \circ_B f(x) = f(x \circ_A x) = f(y \circ_A y) = f(y) \circ_B f(y)$, relation which means $H'_{p'(f(x))} = f(H_i) = H'_{p'(f(y'))}$, whence $p'(f(x)) = p'(f(y))$, since for $\{r_1, r_2\} \subset I'$, $r_1 \neq r_2, H'_{r_1} \cap H'_{r_2} = \emptyset$.

We define the function $\varphi : I \longrightarrow I'$ as it follows:

$$\text{for } \ i = p(x), x \in H, \varphi(i) = p'(f(x)).$$

We prove that φ is an injective function. We suppose that there exist $j_1, j_2 \in I$, $j_1 \neq j_2$, such that $\varphi(j_1) = \varphi(j_2)$. It follows $f(H_{j_1}) = f(H_{j_2})$, which is absurd since $H_{j_1} \cap H_{j_2} = \emptyset$ and f is a bijective function.

φ is also surjective. Indeed, since $f(H) = H$, we have:

$$H = \bigcup_{i' \in I'} H'_{i'} = f\left(\bigcup_{i \in I} H_i\right) = \bigcup_{i \in I} f(H_i) = \bigcup_{i \in I} H'_{\varphi(i)} = \bigcup_{i' \in \mathrm{Im}\varphi} H'_{i'},$$

so $I' = \mathrm{Im}\varphi$.

Therefore φ is a bijection and $\mid I \mid = \mid I' \mid$.

f is an isomorphism, so for any $i \in I, \mid f(H_i) \mid = \mid H_i \mid$, that is, for any $i \in I, a'_{\varphi(i)} = \mid H'_{\varphi(i)} \mid = \mid H_i \mid = a_i$.

We prove now the strict monotony of φ. We shall use the following notations: $u \wedge v = \min\{\mathrm{u,v}\}$, $u \vee v = \max\{\mathrm{u,v}\}$;
$\forall\, (i,j) \in I^2, [i \wedge j, i \vee j] = \{t \in I \mid i \wedge j \leq t \leq i \vee j\}$.

Let $(i,j) \in I^2, i < j$ and let us consider $x \in H_i$ and $y \in H_j$. From $f(x \circ_A y) = f(x) \circ_B f(y)$ it follows

$$f\left(\bigcup_{i \leq r \leq j} H_r\right) = \bigcup_{\varphi(i) \wedge \varphi(j) \leq k \leq \varphi(i) \vee \varphi(j)} H'_k,$$

whence

$$\bigcup_{i \leq r \leq j} H'_{\varphi(r)} = \bigcup_{\varphi(i) \wedge \varphi(j) \leq k \leq \varphi(i) \vee \varphi(j)} H'_k;$$

therefore, $\{\varphi(r) \mid i \leq r \leq j\} = \{k \in I \mid \varphi(i) \wedge \varphi(j) \leq k \leq \varphi(i) \vee \varphi(j)\}$, that is

$$(*) \qquad \forall\, (i,j) \in I^2, i < j, \varphi([i,j]) = [\varphi(i) \wedge \varphi(j), \varphi(i) \vee \varphi(j)].$$

We have $\varphi(i) \neq \varphi(j)$, since $i < j$ and φ is a bijection; so, there are two possibilities: $\varphi(i) < \varphi(j)$ or $\varphi(i) > \varphi(j)$.

For $\varphi(i) < \varphi(j)$, φ is strictly increasing on $[i,j]$. Indeed, let us observe that:

1) if $[i,j] = \{i,j\}$, that is obviously;

2) if there exists $r \in I$ such that $i < r < j$, then
$\varphi(i) = \varphi(i) \wedge \varphi(j) < \varphi(r) < \varphi(i) \vee \varphi(j) = \varphi(j)$, since $(*)$.

3) if there exists $(r,s) \in I^2$, such that $i < r < s < j$ we obtain:
$\varphi([i,r]) = [\varphi(i) \wedge \varphi(r), \varphi(i) \vee \varphi(r)]$; so, using 2), $\varphi(i) = \varphi(i) \wedge \varphi(r) <$
$< \varphi(s) < \varphi(i) \vee \varphi(r) = \varphi(r) < \varphi(j)$.

For $\varphi(j) < \varphi(i)$, we can prove that φ is strictly decreasing in a similar manner.

Moreover, we shall prove that for any $(i, j) \in I^2$, such that $i < j$, we have two situations:

1°. If $\varphi(i) < \varphi(j)$, then φ is strictly increasing on I;

2°. If $\varphi(j) < \varphi(i)$, then φ is strict decreasing on I.

Indeed, in the first situation, we have already seen that φ is strictly increasing on $[i, j]$. Let l be an arbitrary element of I, such that $j < l$. So, $\varphi(j) \neq \varphi(l)$; if $\varphi(l) < \varphi(j)$, then $\varphi(l) < \varphi(i)$.

Indeed, $\varphi(l) \notin [\varphi(i), \varphi(j)] = \varphi([i, j])$, since φ is one-to-one.

But, from $\varphi(l) < \varphi(i)$, we obtain that φ is strictly decreasing on $[i, l]$, as it follows from the second case. So that, φ is strictly decreasing on $[i, j]$ too, which is not true. Therefore, for any $l \in I, j < l, \varphi(j) < \varphi(l)$, that is, φ is strictly increasing on $[j, l], \forall\, l > j$.

Similarly, we can prove that φ is strictly increasing on $[l, i]$, for any $l \in I, l < i$. In conclusion, φ is strictly increasing on I.

Analogously it follows 2°.

Now we prove the implication " $\Longrightarrow$ ".

For any $i \in I$, we have $\mid K_i \mid = \mid H_i \mid = a_i = a'_{\varphi(i)} = \mid H'_{\varphi(i)} \mid = \mid K'_{\varphi(i)} \mid$, so we can suppose $K_i = K'_{\varphi(i)}$.

Let us define $f : (H, \circ_A) \longrightarrow (H, \circ_B)$ in this way: for any $i \in I$, for any $k \in K_i, f(x_{k,i}) = x'_{k,\varphi(i)}$. Hence, f is a bijection.

We verify that f is a homomorphism, too.

For any $(i, j) \in I^2$ and for any $(x, y) \in H_i \times H_j$, there exists $k \in K_i$ and there exists $h \in K_j$ such that $x = x_{k,i}$ and $y = y_{h,j}$. We have:

$$f(x) \circ_B f(y) = f(x_{k,i}) \circ_B f(y_{h,j}) = x'_{k,\varphi(i)} \circ_B y'_{h,\varphi(j)} = \bigcup_{\varphi(i)\wedge\varphi(j)\leq t\leq\varphi(i)\vee\varphi(j)} H'_t;$$

$$f(x \circ_A y) = f\left(\bigcup_{i\wedge j\leq k\leq i\vee j} H_k\right) = \bigcup_{i\wedge j\leq k\leq i\vee j} H'_{\varphi(k)}.$$

If φ is strictly increasing, then

$$\varphi(i \wedge j) = \varphi(i) \wedge \varphi(j) \text{ and } \varphi(i \vee j) = \varphi(i) \vee \varphi(j);$$

φ is also a bijection, so

$$f(x \circ_A y) = \bigcup_{\varphi(i\wedge j)\leq\varphi(k)\leq\varphi(i\vee j)} H'_{\varphi(k)} = \bigcup_{\varphi(i)\wedge\varphi(j)\leq s\leq\varphi(i)\vee\varphi(j)} H'_s,$$

whence $f(x \circ_A y) = f(x) \circ_B f(y)$.

If φ is strictly decreasing, then

$$\varphi(i \wedge j) = \varphi(i) \vee \varphi(j) \text{ and } \varphi(i) \vee \varphi(j) = \varphi(i) \wedge \varphi(j);$$

φ is also a bijection, so

$$f(x \circ_A y) = \bigcup_{\varphi(i \vee j) \leq \varphi(k) \leq \varphi(i) \vee \varphi(j)} H'_{\varphi(k)} = \bigcup_{\varphi(i) \wedge \varphi(j) \leq s \leq \varphi(i) \vee \varphi(j)} H'_s,$$

whence again $f(x \circ_A y) = f(x) \circ_B f(y)$.

Therefore, we have obtained that f is a homomorphism and now the theorem is proved.

3.3 The sequence of join spaces associated with a hypergroupoid

A third connection between the class of hypergroups and the class of fuzzy sets has been established by Corsini in [17], which opened a new field of research and its study represents the aim of this thesis. Every hypergroup H determines a sequence of fuzzy sets and of join spaces associated to H. M.Ştefănescu, P.Corsini, V.Leoreanu and I.Cristea have proved some properties of this sequence and have studied it in some particular cases: for the i.p.s. hypergroups of order less or equal to 7, 1-hypergroups, finite complete hypergroups or for hypergroups obtained from a rough set.

3.3.1 The construction of the sequence of join spaces and fuzzy sets

In the following, we denote by $\langle H, \circ \rangle$ a hypergroupoid.

For any $(x, y) \in H^2$ and for any $u \in H$, we consider

$$(\omega) \qquad \begin{aligned} &\mu_{x,y}(u) = 0 \qquad \text{iff } u \notin x \circ y \\ &\mu_{x,y}(u) = \frac{1}{|x \circ y|} \quad \text{iff } u \in x \circ y \\ &A(u) = \sum_{(x,y) \in H^2} \mu_{x,y}(u) \\ &Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\} \\ &q(u) = |Q(u)| \\ &\widetilde{\mu}(u) = A(u)/q(u). \end{aligned}$$

Then we obtain a join space ${}^1H$ as follows (see [9]):

$$\forall (x, y) \in H^2, \quad x \circ_1 y = \{z \mid \widetilde{\mu}(x) \wedge \widetilde{\mu}(y) \leq \widetilde{\mu}(z) \leq \widetilde{\mu}(x) \vee \widetilde{\mu}(y)\}.$$

By using the same procedure as in (ω), from ${}^1H$ we can obtain a membership function $\widetilde{\mu}_1$ and the associated join space ${}^2H$ and so on. A sequence of fuzzy sets and join spaces $(\langle {}^rH, \circ_r \rangle, \widetilde{\mu}_r)_{r \geq 1}$ is determined in this way. We denote $\widetilde{\mu}_0 = \widetilde{\mu}$,

${}^0H = H$.
Then, for any i, there are r, namely $r = r_i$, and a partition $\pi = \{{}^iC_j\}_{j=1}^r$ of ${}^iH$ such that for any $j \geq 1$ and $x \in {}^iC_j$, ${}^iC_j = \widetilde{\mu}_{i-1}^{-1}(\widetilde{\mu}_{i-1}(x))$ (this means $x, y \in {}^iC_j \Longleftrightarrow \widetilde{\mu}_{i-1}(x) = \widetilde{\mu}_{i-1}(y)$).
For $x \in H$, we denote $\lambda(x) = i_j$, when $x \in {}^iC_j$.
On the set of classes $\{{}^iC_j\}_{j=1}^r$ we define the following ordering relation :
$i_j < i_k$ if for elements $x \in {}^iC_j$ and $y \in {}^iC_k$, $\widetilde{\mu}_{i-1}(x) < \widetilde{\mu}_{i-1}(y)$ (therefore $\lambda(x) < \lambda(y)$).

We shall use some notations, for all j, s :
$k_j = \mid {}^iC_j \mid$, ${}_sC = \bigcup_{1\leq j\leq s} {}^iC_j$, ${}^sC = \bigcup_{s\leq j\leq r} {}^iC_j$, ${}_sk = |{}_sC|$, ${}^sk = \mid {}^sC \mid$ and
$I(i,j) = \{v \in I_{m+1}^* \mid i \wedge j \leq v \leq i \vee j\}$.
With any ordered chain $({}^iC_{j_1}, {}^iC_{j_2}, ..., {}^iC_{j_r})$ we may associate an ordered r-tuple $(k_{j_1}, k_{j_2}, ..., k_{j_r})$, where $k_{j_l} = \mid {}^iC_{j_l} \mid$, for all l, $1 \leq l \leq r$.

In particular, with any hypergroupoid H we can associate an ordered r-tuple $(k_1, k_2, ..., k_r)$.

Theorem 3.3.1. (see [17]) *For any $z \in C_s$, $s = 1, 2, ..., r$,*

$$\widetilde{\mu}_r(z) = \frac{2\sum\limits_{\substack{i\leq s\leq j\\ i\neq j}}^{i,j}\left(\frac{k_ik_j}{\sum\limits_{i\leq t\leq j} k_t}\right) + \frac{k_s^2}{k_s}}{2{}_sk^sk - k_s^2}.$$

Proof. Indeed, for any $(x, y) \in H^2$, we have

$$x \circ_{{}^{r-1}\mu} y = \bigcup_{t\in I(\lambda(x),\lambda(y))} C_t.$$

If $x \in C_s$, then
$q(x) = 2{}_sk^sk - k_s^2$, $A(x) = \sum\limits_{a\circ_{{}^{r-1}\mu}b\ni x} \mu_{a,b}(x)$, where $\mu_{a,b}(x) = \dfrac{1}{\mid a \circ_{{}^{r-1}\mu} b \mid}$.

Setting, for any $z \in H$ and for any $(x, y) \in H^2$,

$$1_{x,y}(z) = 1 \Longleftrightarrow z \in x \circ_{{}^{r-1}\mu} y$$

$$1_{x,y}(z) = 0 \Longleftrightarrow z \notin x \circ_{{}^{r-1}\mu} y$$

we obtain

$${}^r\mu(z) = \frac{\sum\limits_{(x,y)\in H^2}\left(\frac{1_{x,y}(z)k_{\lambda(x)}k_{\lambda(y)}}{\sum\limits_{t\in I(\lambda(x),\lambda(y))} k_t}\right)}{2{}_sk^sk - k_s^2}.$$

From this, the theorem follows straightforward.

Remark 3.3.2. It is easy to observe that if with a hypergroupoid H_1 we associate the r-tuple $(k_1, k_2, ..., k_r)$ and to an other hypergroupoid H_2 we associate the r-tuple $(k_r, k_{r-1}, ..., k_2, k_1)$, then the join spaces ${}^1H_1$ and ${}^1H_2$ associated with H_1 and, respectively, with H_2 are identical.

Theorem 3.3.3. (see [17]) *Let $\langle H, \circ\rangle$ be a hypergroup, $\widetilde{\mu} : H \longrightarrow [0,1]$ a fuzzy sets on H and ${}^1H$ the associated join space. Let $R_{\widetilde{\mu}}$ be the equivalence defined by*

$$xR_{\widetilde{\mu}}y \Longleftrightarrow \widetilde{\mu}(x) = \widetilde{\mu}(y).$$

Then $R_{\widetilde{\mu}}$ is a regular equivalence and a congruence in ${}^1H$.

Proof. For any $(a, b) \in H^2$, $a \circ_{\widetilde{\mu}} b = \{x \in H \mid \widetilde{\mu}(x) \in [\widetilde{\mu}(a), \widetilde{\mu}(b)]\}$; if $aR_{\widetilde{\mu}}a'$ and $bR_{\widetilde{\mu}}b'$ then $a' \circ_{\widetilde{\mu}} b = a \circ_{\widetilde{\mu}} b = a \circ_{\widetilde{\mu}} b'$, so $R_{\widetilde{\mu}}$ is regular in ${}^1H$.

Set now $z'R_{\widetilde{\mu}}z$, where $z \in u \circ_{\widetilde{\mu}} v$, with $uR_{\widetilde{\mu}}x$, $vR_{\widetilde{\mu}}y$. Then we have $\widetilde{\mu}(z) = \widetilde{\mu}(z') \in [\widetilde{\mu}(u), \widetilde{\mu}(v)] = [\widetilde{\mu}(x), \widetilde{\mu}(y)]$ so, $R_{\widetilde{\mu}}(z) \subset R_{\widetilde{\mu}}(x) \circ_{\widetilde{\mu}} R_{\widetilde{\mu}}(y)$ whence $R_{\widetilde{\mu}}$ is a congruence.

3.3.2 Properties of the membership function $\widetilde{\mu}_i$

It is easy to prove the following remark:

Remark 3.3.4. From the relation (ω), for each $i \geq 0$ and $u \in H$ and for $\widetilde{\mu}_i(u)$, we obtain

$$A(u) = \frac{p_1}{m_1} + \frac{p_2}{m_2} + \cdots + \frac{p_s}{m_s},$$

with $m_1 < m_2 < ... < m_s$ and $p_1, p_2, ..., p_s \geq 1$,

$$q(u) = p_1 + p_2 + ... + p_s.$$

Proposition 3.3.5. *For any $u \in H$ and $i \geq 0$, $\widetilde{\mu}_i(u) = 1$ if and only if for all $(x, y) \in H^2$ such that $u \in x \circ_i y$, $\mid x \circ_i y \mid = 1$.*

Proof. Assume $\widetilde{\mu}_i(u) = 1$; then $A(u) = q(u)$.
By using the Remark 3.3.4, we get the relation:

$$p_1(1 - \frac{1}{m_1}) + p_2(1 - \frac{1}{m_2}) + ... + p_s(1 - \frac{1}{m_s}) = 0$$

which implies that, for any $j, 1 \leq j \leq s, p_j(1 - \dfrac{1}{m_j}) = 0$.
But $p_j \neq 0$, for all $1 \leq j \leq s$, therefore $m_j = 1$, for all $1 \leq j \leq s$.
As $m_1 < m_2 < ... < m_s$, it results that $s = 1, m_1 = 1$ and for all $(x, y) \in H^2$ with $u \in x \circ_i y$, $\mid x \circ_i y \mid = 1$.

Conversely, suppose that for any $(x, y) \in H^2$ with $u \in x \circ_i y$, $\mid x \circ_i y \mid = 1$; then $\mu_{x,y}(u) = 1$ and

$$A(u) = \sum_{(x,y)\in^2 H} \mu_{x,y}(u) = p, \text{ where } p = \mid \{(x, y) \in^2 H \mid u \in x \circ_i y\} \mid = q(u),$$

thus $\widetilde{\mu}_i(u) = 1$.

This result allows us to prove the following proposition.

Proposition 3.3.6. *For any $u \in H$ and $i \geq 1$, $\widetilde{\mu}_i(u) \neq 1$.*

Proof. For each $u \in H$ and $i \geq 1$ there are only two possibilities:

(i) For any $v \in H, v \neq u$, $\widetilde{\mu}_i(u) = \widetilde{\mu}_i(v)$.
In this case one obtains that ${}^{i+1}H$ is a total hypergroup (this means that for any $(x, y) \in H^2, x \circ_{i+1} y = H$) and then $\widetilde{\mu}_{i+1}(u) = \frac{1}{n}$, where $\mid H \mid = n$.

(ii) There exists $v \in H, v \neq u$ such that $\widetilde{\mu}_i(u) \neq \widetilde{\mu}_i(v)$. Then $\widetilde{\mu}_i(u) < \widetilde{\mu}_i(v)$ or $\widetilde{\mu}_i(u) > \widetilde{\mu}_i(v)$ which implies that $u, v \in u \circ_{i+1} v$, so $\mid u \circ_{i+1} v \mid \geq 2$ and by the Proposition 3.3.5, $\widetilde{\mu}_{i+1}(u) \neq 1$.

Remark 3.3.7. There are hypergroups H such that

(i) $A(x) = A(y)$ and $q(x) \neq q(y)$, for some $x, y \in H$.

(ii) $A(x) \neq A(y)$ and $q(x) = q(y)$, for some $x, y \in H$.

In the both cases $\widetilde{\mu}(x) \neq \widetilde{\mu}(y)$.
An example for the case (i) is the hypergroup H_1 and the case (ii) is illustrated by the hypergroup H_2 given bellow.

H_1	x_1	x_2	x_3	x_4
x_1	x_1, x_2	x_1, x_2	$H \setminus \{x_4\}$	H
x_2		x_1, x_2	$H \setminus \{x_4\}$	H
x_3			x_3	x_3, x_4
x_4				x_4

where $A(x_1) = A(x_3) = \frac{13}{3}$, $q(x_1) = 12$, $q(x_3) = 11$ and by consequence $\widetilde{\mu}(x_1) = 0,36 \neq 0,39 = \widetilde{\mu}(x_3)$.

H_2	x_1	x_2	x_3	x_4	x_5
x_1	x_1	x_1, x_2	$H \setminus \{x_5\}$	$H \setminus \{x_5\}$	H
x_2		x_2	x_2, x_3, x_4	x_2, x_3, x_4	$H \setminus \{x_1\}$
x_3			x_3, x_4	x_3, x_4	x_3, x_4, x_5
x_4				x_3, x_4	x_3, x_4, x_5
x_5					x_5

where $q(x_1) = q(x_5) = 9$, $A(x_1) = \dfrac{17}{5}$, $A(x_5) = \dfrac{97}{30}$ and by consequence $\widetilde{\mu}(x_1) = 0,38 \neq 0,36 = \widetilde{\mu}(x_5)$.

Remark 3.3.8. Assume that the hypergroupoid H has the decomposition $H = \bigcup_{i=1}^{r} C_i$, where $C_i = \widetilde{\mu}^{-1}(\widetilde{\mu}(x))$, for any $x \in H$.
If $x \in C_i$ and $y \in C_j$, $i \neq j$, with $| C_i |=| C_j |$, then $q(x) = q(y)$ if and only if ${}_ik = k^j$ if and only if ${}^ik = k_j$, as we can easily see from the previous notations and definitions.
If $i = 1$ and $j = r$, that is $x \in C_1$ and $y \in C_r$, then $\widetilde{\mu}(x) < \widetilde{\mu}(u)$, for any $u \in H \backslash C_1$ and $\widetilde{\mu}(y) > \widetilde{\mu}(v)$, for any $v \in H \setminus C_r$. Therefore ${}_1k = k_1 = k_r = {}^rk$. It follows that $q(x) = q(y)$.

3.3.3 Fuzzy grade of the hypergroups

Definition 3.3.9. (see [19]) A hypergroupoid H has the *fuzzy grade* m, $m \in \mathbb{N}^*$, and we write $f.g.(H) = m$ if, for any i, $0 \leq i < m$, the join spaces ${}^iH$ and ${}^{i+1}H$ associated with H are not isomorphic (where ${}^0H = H$) and, for any s, $s > m$, ${}^sH$ is isomorphic with ${}^mH$.
We say that the hypergroupoid H has the *strong fuzzy grade* m and we write $s.f.g.(H) = m$ if $f.g.(H) = m$ and, for all $s, s > m$, ${}^sH = {}^mH$.

First we determine the fuzzy grade of a hypergroupoid H for which $|H/R_{\widetilde{\mu}}| \in \{2, 3\}$.

Case I. If the hypergroupoid H has only two classes of equivalence, that is $H = C_1 \bigcup C_2$, where $|C_1| = k_1$ and $|C_2| = k_2$, $k_1 + k_2 = n = |H|$, then there are only two possibilities:

a) $k_1 = k_2$ and then the join space ${}^1H$ is a total hypergroup and therefore $s.f.g.(H) = 2$. Indeed, by the Theorem 3.3.1, $\widetilde{\mu}_1(x) = \widetilde{\mu}_1(y)$, for any $x \in C_1$ and for any $y \in C_2$.

b) $k_1 < k_2$ with $\widetilde{\mu}_1(x) = \dfrac{k_1 + 3k_2}{2n^2}$, for any $x \in C_1$ and similarly, for any $y \in C_2$, $\widetilde{\mu}_1(y) = \dfrac{k_2 + 3k_1}{2n^2}$; this means $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(y)$ and thus ${}^2H$ is isomorphic with ${}^1H$, $s.f.g.(H) = 1$.

Case II. If the hypergroupoid H has three classes of equivalence, that is $H = C_1 \bigcup C_2 \bigcup C_3$, where $|C_1| = k_1$, $|C_2| = k_2$ and $|C_3| = k_3$ with $k_1 + k_2 + k_3 =$
$= n = |H|$, then there are the following possibilities:

a) $k_1 = k_2 = k_3 = k$; by the Theorem 3.3.1 we have for $x \in C_1$, $A(x) = \dfrac{8}{3k}, q(x) = 5k^2$ and $\widetilde{\mu}_1(x) = \dfrac{8}{15k}$;

for $y \in C_2$, $A(y) = \dfrac{11}{3k}, q(y) = 7k^2$ and $\widetilde{\mu}_1(y) = \dfrac{11}{21k}$;

for $z \in C_3$, $A(z) = \dfrac{8}{3k}, q(x) = 5k^2$ and $\widetilde{\mu}_1(z) = \dfrac{8}{15k}$.

Therefore, $\widetilde{\mu}_1(x) = \widetilde{\mu}_1(z) > \widetilde{\mu}_1(y)$ and with the join space ${}^1H$ we associate the pair $(k, 2k)$. By the Case I, we conclude that $s.f.g.(H) = 2$.

b) $k_1 = k_3 \neq k_2$; using again the Theorem 3.3.1 we obtain $\widetilde{\mu}_1(x) = \widetilde{\mu}_1(z) \neq \neq \widetilde{\mu}_1(y)$ as in the case a) and therefore $f.g.(H) = 2$.

c) $k_1 < k_2 = k_3$; by the Theorem 3.3.1 we obtain

$$\text{for } \; x \in C_1, A(x) = k_1 + 2\frac{k_1k_2}{k_1+k_2} + 2\frac{k_1k_2}{k_1+2k_2}, q(x) = k_1(k_1+4k_2)$$

$$\text{and } \; \widetilde{\mu}_1(x) = \frac{1 + 2k_2\left(\dfrac{1}{k_1+k_2} + \dfrac{1}{k_1+2k_2}\right)}{k_1+4k_2};$$

$$\text{for } \; y \in C_2, A(y) = 2k_2\left(1 + \frac{k_1}{k_1+k_2} + \frac{k_1}{k_1+2k_2}\right), q(y) = k_2(4k_1+3k_2)$$

$$\text{and } \; \widetilde{\mu}_1(y) = \frac{2 + 2k_1\left(\dfrac{1}{k_1+k_2} + \dfrac{1}{k_1+2k_2}\right)}{4k_1+3k_2};$$

$$\text{for } \; z \in C_3, A(z) = 2k_2\left(1 + \frac{k_1}{k_1+2k_2}\right), q(z) = k_2(2k_1+3k_2)$$

$$\text{and } \; \widetilde{\mu}_1(z) = \frac{2 + 2k_1\dfrac{1}{k_1+2k_2}}{2k_1+3k_2}.$$

We verify if $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(y)$; it is equivalent with

$$\frac{1 + 2k_2\left(\dfrac{1}{k_1+k_2} + \dfrac{1}{k_1+2k_2}\right)}{k_1+4k_2} > \frac{2 + 2k_1\left(\dfrac{1}{k_1+k_2} + \dfrac{1}{k_1+2k_2}\right)}{4k_1+3k_2} \Longleftrightarrow$$

$$8k_2^3 - 2k_1^3 - 5k_1^2k_2 + k_1k_2^2 > 0 \Longleftrightarrow$$

$$5k_2(k_2^2 - k_1^2) + 3k_2^3 - 2k_1^3 + k_1k_2^2 > 0 \; \text{ which is true for } \; k_1 < k_2.$$

Now we verify if $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(z)$; that is

$$\frac{1 + 2k_2\left(\dfrac{1}{k_1+k_2} + \dfrac{1}{k_1+2k_2}\right)}{k_1+4k_2} > \frac{2 + 2k_1\dfrac{1}{k_1+2k_2}}{2k_1+3k_2} \Longleftrightarrow$$

$$8k_2^3 - 2k_1^3 - 7k_1^2 k_2 + k_1 k_2^2 > 0 \Longleftrightarrow$$

$$(k_2 - k_1)(2k_1^2 + 9k_1 k_2 + 8k_2^2) > 0; \quad \text{it is true for} \quad k_1 < k_2.$$

Finally, we verify if $\widetilde{\mu}_1(y) = \widetilde{\mu}_1(z)$, that is

$$\frac{2 + 2k_1\left(\dfrac{1}{k_1 + k_2} + \dfrac{1}{k_1 + 2k_2}\right)}{4k_1 + 3k_2} = \frac{2 + 2k_1\dfrac{1}{k_1 + 2k_2}}{2k_1 + 3k_2} \Longleftrightarrow$$

$$2k_1^2 + k_1 k_2 - 2k_2^2 = 0.$$

We consider this equation as an equation of second grade in $k_1 \in \mathbb{N}^*$. The solutions are $\dfrac{-k_2 \pm \sqrt{17}k_2}{2}$ which are not natural numbers, so that $\widetilde{\mu}_1(y) \neq \widetilde{\mu}_1(z)$.
In conclusion, $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(y)$ and $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(z)$, with $\widetilde{\mu}_1(y) \neq \widetilde{\mu}_1(z)$; therefore to the join space ${}^1H$ we associate the triple (k_2, k_2, k_1) and, by consequence, the join spaces ${}^2H$ and ${}^1H$ associated with H are isomorphic and thus $s.f.g.(H) = 1$.

d) $k_1 = k_2 < k_3$; again by the same theorem one obtains that

$$\text{for} \quad x \in C_1, \widetilde{\mu}_1(x) = \frac{2\left(1 + \dfrac{k_3}{2k_1 + k_3}\right)}{3k_1 + 2k_3};$$

$$\text{for} \quad y \in C_2, \widetilde{\mu}_1(y) = \frac{2\left(1 + \dfrac{k_3}{k_1 + k_3} + \dfrac{k_3}{2k_1 + k_3}\right)}{3k_1 + 4k_3};$$

$$\text{for} \quad z \in C_3, \widetilde{\mu}_1(z) = \frac{1 + 2k_1\left(\dfrac{1}{k_1 + k_3} + \dfrac{1}{2k_1 + k_3}\right)}{4k_1 + k_3}.$$

It is simple to verify that $\widetilde{\mu}_1(x) \neq \widetilde{\mu}_1(y)$ and $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(z)$.

Now we see when $\widetilde{\mu}_1(y) > \widetilde{\mu}_1(z)$; this is equivalent with

$$P = 2k_3^3 - 8k_1^3 - k_1^2 k_3 + 5k_1 k_3^2 > 0.$$

If $P > 0$, then $\widetilde{\mu}_1(z) < \widetilde{\mu}_1(y)$ and $\widetilde{\mu}_1(z) < \widetilde{\mu}_1(x)$, with $\widetilde{\mu}_1(x) \neq \widetilde{\mu}_1(z)$. As in the previous case, we have $s.f.g.(H) = 1$.

If $P < 0$, then $\widetilde{\mu}_1(y) < \widetilde{\mu}_1(z) < \widetilde{\mu}_1(x)$ and the triple associated with the join space ${}^1H$ is (k_1, k_3, k_1). By the case a), with the join space ${}^2H$ we associate the pair $(2k_1, k_3)$, with $2k_1 \neq k_3$ (if $2k_1 = k_3$, then $P > 0$) and thus $s.f.g.(H) = 3$. For example, if $k_1 = k_2 = 10$ and $k_3 = 11$, then $P < 0$ and $s.f.g.(H) = 3$.

e) $k_1 \neq k_2 \neq k_3$; we have two possibilities: $k_1 < k_2 < k_3$ or $k_1 < k_3 < k_2$. Using the Theorem 3.3.1 we obtain, in the both situations,

$$\text{for } \ x \in C_1, \widetilde{\mu}_1(x) = \frac{1 + \dfrac{2k_2}{k_1+k_2} + \dfrac{2k_3}{k_1+k_2+k_3}}{k_1+2k_2+2k_3};$$

$$\text{for } \ y \in C_2, \widetilde{\mu}_1(y) = \frac{k_2 + 2\dfrac{k_1k_2}{k_1+k_2} + 2\dfrac{k_1k_3}{k_1+k_2+k_3} + 2\dfrac{k_2k_3}{k_2+k_3}}{2k_1k_2+2k_1k_3+k_2^2+2k_2k_3};$$

$$\text{for } \ z \in C_3, \widetilde{\mu}_1(z) = \frac{1 + \dfrac{2k_1}{k_1+k_2+k_3} + \dfrac{2k_2}{k_2+k_3}}{2k_1+2k_2+k_3}.$$

It is simple to verify that $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(y)$ and $\widetilde{\mu}_1(x) > \widetilde{\mu}_1(z)$, for any $x \in C_1, y \in C_2, z \in C_3$. We can not say that for any triple (k_1, k_2, k_3) there is the same relation between $\widetilde{\mu}_1(y)$ and $\widetilde{\mu}_1(z)$, because in the first situation,

for $(k_1, k_2, k_3) = (10, 11, 12)$ we have $\widetilde{\mu}_1(y) > \widetilde{\mu}_1(z)$;

for $(k_1, k_2, k_3) = (20, 21, 22)$ we have $\widetilde{\mu}_1(y) < \widetilde{\mu}_1(z)$.

Similarly, in the second situation,

for $(k_1, k_2, k_3) = (1, 3, 2)$ we have $\widetilde{\mu}_1(y) < \widetilde{\mu}_1(z)$;

for $(k_1, k_2, k_3) = (10, 20, 19)$ we have $\widetilde{\mu}_1(y) > \widetilde{\mu}_1(z)$.

So, with the join space ${}^1H$ one associates the triple (k_3, k_2, k_1) and then $s.f.g.(H) = 1$, or the triple (k_2, k_3, k_1); in this situation, with the join space ${}^2H$ one associates the triple (k_2, k_3, k_1) and thus $s.f.g.(H) = 2$ or the triple (k_3, k_2, k_1) and therefore $f.g.(H) = 2$.

It remains an open problem to see if and when $\widetilde{\mu}_1(y) = \widetilde{\mu}_1(z)$. In this case the join spaces ${}^1H$ and ${}^2H$ associated with H are not isomorphic and therefore $f.g.(H) = 2$.

Theorem 3.3.10. *Let H be the hypergroupoid $H = \{a_0, a_1, ..., a_n\}$ with $n = 2p+1$, $p \in \mathbb{N} \setminus \{0\}$, endowed with the fuzzy set $\widetilde{\mu}$ such that*

$$\widetilde{\mu}(a_0) < \widetilde{\mu}(a_1) < ... < \widetilde{\mu}(a_p) < ... < \widetilde{\mu}(a_n).$$

Then we obtain that

$$\begin{array}{ccccccc} \widetilde{\mu}_1(a_p) < & \widetilde{\mu}_1(a_{p-1}) < & ... < & \widetilde{\mu}_1(a_1) < & \widetilde{\mu}_1(a_0) \\ \| & \| & & \| & \| \\ \widetilde{\mu}_1(a_{p+1}) & \widetilde{\mu}_1(a_{p+2}) & ... & \widetilde{\mu}_1(a_{n-1}) & \widetilde{\mu}_1(a_n). \end{array}$$

Proof. The join space ${}^1H$ associated is

${}^1H$	a_0	a_1	a_2	$\cdots$	a_p	a_{p+1}	$\cdots$	a_{n-1}	a_n
a_0	a_0	a_0, a_1	a_0, a_1, a_2	$\cdots$	$a_0 \to a_p$	$a_0 \to a_{p+1}$	$\cdots$	H_{a_n}	H
a_1		a_1	a_1, a_2	$\cdots$	$a_1 \to a_p$	$a_1 \to a_{p+1}$	$\cdots$	$a_1 \to a_{n-1}$	H_{a_o}
$\vdots$			$\ddots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_p					a_p	a_p, a_{p+1}	$\cdots$	$a_p \to a_{n-1}$	$a_p \to a_n$
a_{p+1}						a_{p+1}	$\cdots$	$a_{p+1} \to a_{n-1}$	$a_{p+1} \to a_n$
$\vdots$							$\ddots$	$\cdots$	$\cdots$
a_{n-1}								a_{n-1}	a_{n-1}, a_n
a_n									a_n

where $H_{a_i} = H \backslash \{a_i\}$ and $a_i \to a_j = \{a_i, a_{i+1}, \ldots, a_j\}$ with $i < j$.

For $i \in \{0, 1, \ldots, n\}$ one calculates $\widetilde{\mu}_1(a_i)$ and one obtains:

$$A(a_0) = 1 + \frac{2}{2} + \frac{2}{3} + \ldots + \frac{2}{n+1}\ ,\ q(a_0) = 1 + 2n$$

$$A(a_n) = 1 + \frac{2}{2} + \frac{2}{3} + \ldots + \frac{2}{n+1}\ ,\ q(a_n) = 1 + 2n$$

$$A(a_1) = 1 + \frac{4}{2} + \frac{4}{3} + \ldots + \frac{4}{n} + \frac{2}{n+1}\ ,\ q(a_1) = 4n - 1$$

$$A(a_{n-1}) = 1 + \frac{4}{2} + \frac{4}{3} + \ldots + \frac{4}{n} + \frac{2}{n+1}\ ,\ q(a_{n-1}) = 4n - 1$$

$$A(a_2) = 1 + \frac{4}{2} + \frac{6}{3} + \frac{6}{4} + \ldots + \frac{6}{n-1} + \frac{4}{n} + \frac{2}{n+1}\ ,\ q(a_2) = 6n - 7$$

$$A(a_{n-2}) = 1 + \frac{4}{2} + \frac{6}{3} + \frac{6}{4} + \ldots + \frac{6}{n-1} + \frac{4}{n} + \frac{2}{n+1},\ q(a_{n-2}) = 6n - 7$$

More generally, for $1 \leq j \leq p$,

$$\begin{aligned} A(a_j) = {} & 1 + \frac{4}{2} + \frac{6}{3} + \ldots + \frac{2(j+1)}{j+1} + \frac{2(j+1)}{j+2} + \ldots + \\ & + \frac{2(j+1)}{n-j+1} + \frac{2j}{n-j+2} + \frac{2(j-1)}{n-j+3} + \ldots + \frac{4}{n} + \frac{2}{n+1} \\ q(a_j) = {} & 2(j+1)(n-j+1) - 1. \end{aligned}$$

We observe that $\widetilde{\mu}_1(a_j) = \widetilde{\mu}_1(a_{n-j})$ for $0 \leq j \leq p$.

We will prove that, for $0 \leq j \leq p$, it is satisfied the relation $\widetilde{\mu}_1(a_j) < \widetilde{\mu}_1(a_{j-1})$. We have:

$$A(a_{j-1}) = 1 + \frac{4}{2} + \ldots + \frac{2j}{j} + \frac{2j}{j+1} + \ldots + \frac{2j}{n-j+2} + \frac{2(j-1)}{n-j+3} + \frac{2(j-2)}{n-j+4} + \ldots + \frac{4}{n} + \frac{2}{n+1}$$

$$q(a_{j-1}) = 2j(n-j+2) - 1,$$

so

$$A(a_j) = A(a_{j-1}) + 2\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{n-j+1}\right).$$

Thus, we arrive at the following inequality:

$$\frac{A(a_{j-1})}{2nj - 2j^2 + 4j - 1} > \frac{A(a_{j-1}) + 2\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{n-j+1}\right)}{2nj - 2j^2 + 2n + 1} \Longleftrightarrow$$

$$(1) \qquad A(a_{j-1}) > \frac{2nj - 2j^2 + 4j - 1}{n - 2j + 1}\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{n-j+1}\right).$$

Rewriting $A(a_{j-1})$ in the form

$$\begin{aligned} A(a_{j-1}) &= 2j - 1 + 2j\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{n-j+1} + \frac{1}{n-j+2}\right) + \\ &+ 2\left(\frac{j-1}{n-j+3} + \ldots + \frac{2}{n} + \frac{1}{n+1}\right) \end{aligned}$$

it follows that the inequality (1) is equivalent with the following one

$$2j-1+\frac{2j}{n-j+2}+2j\left(\frac{1}{j+1}+\ldots+\frac{1}{n-j+1}\right)+2\left(\frac{j-1}{n-j+3}+\ldots+\frac{2}{n}+\frac{1}{n+1}\right) >$$

$$> \frac{2nj - 2j^2 + 4j - 1}{n - 2j + 1}\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{n-j+1}\right) \Longleftrightarrow$$

$$(2) \qquad \begin{aligned} &\frac{2nj - 2j^2 + 7j - n - 2}{n - j + 2} + 2\left(\frac{j-1}{n-j+3} + \ldots + \frac{2}{n} + \frac{1}{n+1}\right) > \\ &> \frac{2j^2 + 2j - 1}{n - 2j + 1}\left(\frac{1}{j+1} + \frac{1}{j+2} + \ldots + \frac{1}{n-j+1}\right), \end{aligned}$$

with $n - 1 \geq 2j$.

We shall prove the relation by induction in j.
For $j = 3$ we obtain

$$\frac{5n+1}{n-1} + 2\left(\frac{2}{n} + \frac{1}{n+1}\right) > \frac{23}{n-5}\left(\frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n-2}\right) \Longleftrightarrow$$

$$\frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n-2} < \frac{(n-5)(5n^3 + 12n^2 - n - 4)}{23n(n^2-1)}.$$

It is obvious (by induction in n) that, for $n \geq 6$, the relation is true.

Now let's suppose that the relation (2) is true for j and let's prove it for $j+1$:

$$\frac{2nj+n-2j^2+3j+3}{n-j+1}+2\left(\frac{j}{n-j+2}+\frac{j-1}{n-j+3}+\ldots+\frac{1}{n+1}\right)>$$
$$>\frac{2j^2+6j+3}{n-2j-1}\left(\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{n-j}\right).$$

Using the induction hypothesis, one obtains:

$$\frac{2nj+n-2j^2+3j+3}{n-j+1}+2\left(\frac{j}{n-j+2}+\frac{j-1}{n-j+3}+\ldots+\frac{1}{n+1}\right)>\frac{2nj+n-2j^2+3j+3}{n-j+1}+$$

$$+\frac{2j}{n-j+2}+\frac{2j^2+2j-1}{n-2j+1}\left(\frac{1}{j+1}+\ldots+\frac{1}{n-j+1}\right)-\frac{2nj-2j^2+7j-n-2}{n-j+2}.$$

Therefore it remains to verify that

$$\frac{2nj+n-2j^2+3j+3}{n-j+1}-\frac{2nj-2j^2+5j-n-2}{n-j+2}+\frac{2j^2+2j-1}{n-2j+1}\left(\frac{1}{j+1}+\ldots+\frac{1}{n-j+1}\right)>$$
$$>\left(\frac{2j^2+6j+3}{n-2j-1}-\frac{2j^2+2j-1}{n-2j+1}\right)\left(\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{n-j}\right).$$

Calculating, it results that

$$\frac{(2n^3j+2n^3-4n^2j^2+6n^2j+9n^2+2nj^3-12nj^2+3nj+12n+4j^3-8j^2-2j+4)}{(j+1)(n-j+1)(n-j+2)(4nj+4n-4j^2+2)}>$$

$$>\frac{1}{n-2j-1}\left(\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{n-j}\right),\quad \text{for all} \quad j, 3\le j\le p.$$

The first member of the inequality can be made smaller than $\dfrac{n}{3(j+1)}$ and for this reason we must prove that

$$\frac{n}{3(j+1)}>\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{n-j},\qquad \forall j, 3\le j\le p.$$

But,

$$\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{n-j}<\frac{1}{j+2}+\ldots+\frac{1}{3j+2}+\frac{1}{3j+3}+\ldots+\frac{1}{n-j}<$$
$$<\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{3j+2}+\frac{n-4j-2}{3j+3}<\frac{n}{3j+3}$$

if and only if

$$\frac{1}{j+2}+\frac{1}{j+3}+\ldots+\frac{1}{3j+2}<\frac{4j+2}{3j+3}.$$

It is easy to make sure that the last relation is true (it can be verified by induction in j), for every j with $3 \leq j \leq p$.

Finally, we have to prove that

$$\widetilde{\mu}_1(a_2) < \widetilde{\mu}_1(a_1) \text{ and } \widetilde{\mu}_1(a_1) < \widetilde{\mu}_1(a_0).$$

We have that

$$A(a_2) = A(a_1) + 2\left(\frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1}\right)$$

and denoting

$$S = \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1}$$

we have that

$$A(a_2) = A(a_1) + 2S.$$

So

$$\begin{aligned}
\widetilde{\mu}_1(a_2) < \widetilde{\mu}_1(a_1) \Longleftrightarrow \frac{A(a_1)+2S}{6n-7} < \frac{A(a_1)}{4n-1} \Longleftrightarrow \\
(4n-1)S < (n-3)A(a_1) \Longleftrightarrow \\
(n-3)\left(3 + \frac{4}{n} + \frac{2}{n+1} + 4S\right) > (4n-1)S \Longleftrightarrow \\
11S < (n-3) \cdot \frac{3n^2+9n+4}{n(n+1)}.
\end{aligned}$$

But

$$S = \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1} < \frac{1}{3} + \frac{n-4}{4} = \frac{3n-8}{12},$$

then

$$11S < \frac{33n-88}{12} < (n-3)\frac{3n^2+9n+4}{n(n+1)}$$

if and only if

$$\begin{aligned}
&(33n-88)(n^2+n) < (12n-36)(3n^2+9n+4) \Longleftrightarrow \\
&3n^3 + 55n^2 - 188n - 144 > 0.
\end{aligned}$$

The relation is true for $n \geq 6$.

Next, to prove that $\widetilde{\mu}_1(a_1) < \widetilde{\mu}_1(a_0)$ we write

$$A(a_1) = A(a_0) + 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = 2A(a_0) - \frac{n+3}{n+1}.$$

So we have

$$\frac{A(a_0)}{2n+1} > \frac{2A(a_0) - \dfrac{n+3}{n+1}}{4n-1} \Longleftrightarrow$$

$$(3) \qquad A(a_0) < \frac{(2n+1)(n+3)}{n+1}.$$

But

$$A(a_0) = 1 + 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) < 1 + 2 \cdot \frac{n}{2} = n+1,$$

then, to prove (3) we must verify that

$$n + 1 < \frac{(2n+1)(n+3)}{n+1} \Longleftrightarrow n^2 + 5n + 2 > 0,$$

true for every $n \geq 1$.

Remark 3.3.11. If $p = 2r+1$, then we can repeat this process and we obtain

$$\begin{array}{ccccccccc}
\widetilde{\mu}_2(a_n) & & \widetilde{\mu}_2(a_{n-1}) & & \ldots & & \widetilde{\mu}_2(a_{n-r+1}) & & \widetilde{\mu}_2(a_{n-r}) \\
\| & & \| & & & & \| & & \| \\
\widetilde{\mu}_2(a_0) & > & \widetilde{\mu}_2(a_1) & > & \ldots & > & \widetilde{\mu}_2(a_{r-1}) & > & \widetilde{\mu}_2(a_r) \\
\| & & \| & & & & \| & & \| \\
\widetilde{\mu}_2(a_p) & & \widetilde{\mu}_2(a_{p-1}) & & \ldots & & \widetilde{\mu}_2(a_{r+2}) & & \widetilde{\mu}_2(a_{r+1}) \\
\| & & \| & & & & \| & & \| \\
\widetilde{\mu}_2(a_{p+1}) & & \widetilde{\mu}_2(a_{p+2}) & & \ldots & & \widetilde{\mu}_2(a_{n-r-2}) & & \widetilde{\mu}_2(a_{n-r-1})
\end{array}$$

and so on.
Generally, if $n + 1 = 2^s$, we can continue this process till when $\widetilde{\mu}_s(a_k) = \widetilde{\mu}_s(a_\ell)$, for every k and ℓ, and in this way we obtain that the join space ${}^{s+1}H$ associated is a total hypergroup.

Example 3.3.12. For the hypergroupoid $H = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ with $\widetilde{\mu}(a_0) < \widetilde{\mu}(a_1) < \widetilde{\mu}(a_2) < \widetilde{\mu}(a_3) < \widetilde{\mu}(a_4) < \widetilde{\mu}(a_5) < \widetilde{\mu}(a_6) < \widetilde{\mu}(a_7)$ the join space ${}^1H$ associated is

${}^1H$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_0	a_0	a_0, a_1	a_0 a_1, a_2	a_0, a_1 a_2, a_3	a_0 a_1, a_2 a_3, a_4	a_0, a_1 a_2, a_3 a_4, a_5	H_{a_7}	H
a_1		a_1	a_1, a_2	a_1 a_2, a_3	a_1, a_2 a_3, a_4	a_1 a_2, a_3 a_4, a_5	a_1, a_2 a_3, a_4 a_5, a_6	H_{a_0}
a_2			a_2	a_2, a_3	a_2 a_3, a_4	a_2, a_3 a_4, a_5	a_2 a_3, a_4 a_5, a_6	a_2, a_3 a_4, a_5 a_6, a_7
a_3				a_3	a_3, a_4	a_3 a_4, a_5	a_3, a_4 a_5, a_6	a_3 a_4, a_5 a_6, a_7
a_4					$a_4,$	a_4, a_5	a_4, a_5 a_6	a_4, a_5 a_6, a_7
a_5						a_5	a_5, a_6	a_5, a_6 a_7
a_6							a_6	a_6, a_7
a_7								a_7

Therefore one obtains:

$$\widetilde{\mu}_1(a_0) = \widetilde{\mu}_1(a_7) = 0,296$$
$$\widetilde{\mu}_1(a_1) = \widetilde{\mu}_1(a_6) = 0,293$$
$$\widetilde{\mu}_1(a_2) = \widetilde{\mu}_1(a_5) = 0,272$$
$$\widetilde{\mu}_1(a_3) = \widetilde{\mu}_1(a_4) = 0,267$$

whence

$$\begin{array}{ccccccc} \widetilde{\mu}_1(a_3) & < & \widetilde{\mu}_1(a_2) & < & \widetilde{\mu}_1(a_1) & < & \widetilde{\mu}_1(a_0) \\ \| & & \| & & \| & & \| \\ \widetilde{\mu}_1(a_4) & & \widetilde{\mu}_1(a_5) & & \widetilde{\mu}_1(a_6) & & \widetilde{\mu}_1(a_7) \end{array}$$

The join space ${}^2H$ associated is

${}^2H$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_0	a_0, a_7	a_0, a_1 a_6, a_7	a_0, a_1 a_2, a_5 a_6, a_7	H	H	a_0, a_1 a_2, a_5 a_6, a_7	a_0, a_1 a_6, a_7	a_0, a_7
a_1		a_1, a_6	a_1, a_2 a_5, a_6	a_1, a_2 a_3, a_4 a_5, a_6	a_1, a_2 a_3, a_4 a_5, a_6	a_1, a_2 a_5, a_6	a_1, a_6	a_0, a_1 a_6, a_7
a_2			a_2, a_5	a_2, a_3 a_4, a_5	a_2, a_3 a_4, a_5	a_2, a_5	a_1, a_2 a_5, a_6	a_0, a_1 a_2, a_5 a_6, a_7
a_3				a_3, a_4	a_3, a_4	a_2, a_3 a_4, a_5	a_1, a_2 a_3, a_4 a_5, a_6	H
a_4					a_3, a_4	a_2, a_3 a_4, a_5	a_1, a_2 a_3, a_4 a_5, a_6	H
a_5						a_2, a_5	a_1, a_2 a_5, a_6	a_0, a_1 a_2, a_5 a_6, a_7
a_6							a_1, a_6	a_0, a_1 a_6, a_7
a_7								a_0, a_7

Again one obtains that

$$\widetilde{\mu}_2(a_0) = \widetilde{\mu}_2(a_7) = \widetilde{\mu}_2(a_3) = \widetilde{\mu}_2(a_4) = 0,226$$
$$\widetilde{\mu}_2(a_1) = \widetilde{\mu}_2(a_6) = \widetilde{\mu}_2(a_2) = \widetilde{\mu}_2(a_5) = 0,219$$

whence

$$\begin{array}{ccc}
\widetilde{\mu}_2(a_1) & < & \widetilde{\mu}_2(a_0) \\
\| & & \| \\
\widetilde{\mu}_2(a_2) & & \widetilde{\mu}_2(a_3) \\
\| & & \| \\
\widetilde{\mu}_2(a_5) & & \widetilde{\mu}_2(a_4) \\
\| & & \| \\
\widetilde{\mu}_2(a_6) & & \widetilde{\mu}_2(a_7)
\end{array}$$

The join space ${}^3H$ associated is

${}^3H$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_0	a_0, a_3 a_4, a_7	H	H	a_0, a_3 a_4, a_7	H	H	a_0, a_3 a_4, a_7	
a_1		a_1, a_2 a_5, a_6	a_1, a_2 a_5, a_6	H	H	a_1, a_2 a_5, a_6	a_1, a_2 a_5, a_6	H
a_2			a_1, a_2 a_5, a_6	H	H	a_1, a_2 a_5, a_6	a_1, a_2 a_5, a_6	H
a_3				a_0, a_3 a_4, a_7	a_0, a_3 a_4, a_7	H	H	a_0, a_3 a_4, a_7
a_4					a_0, a_3 a_4, a_7	H	H	a_0, a_3 a_4, a_7
a_5						a_1, a_2 a_5, a_6	a_1, a_2 a_5, a_6	H
a_6							a_1, a_2 a_5, a_6	H
a_7								a_0, a_3 a_4, a_7

We notice that $\widetilde{\mu}_3(a_0) = \widetilde{\mu}_3(a_1) = \ldots = \widetilde{\mu}_3(a_7)$ and therefore the join space ${}^4H$ associated is a total hypergroup.

Theorem 3.3.13. *In the same conditions of the Theorem* 3.3.10 *but for* $n = 2p$, $p \in \mathbb{N} \setminus \{0\}$, *we obtain that*

$$\begin{array}{ccccccccc}
\widetilde{\mu}_1(a_p) < & \widetilde{\mu}_1(a_{p-1}) & < & \widetilde{\mu}_1(a_{p-2}) & < \ldots < & \widetilde{\mu}_1(a_1) & < & \widetilde{\mu}_1(a_0) \\
& \| & & \| & & \| & & \| \\
& \widetilde{\mu}_1(a_{p+1}) & & \widetilde{\mu}_1(a_{p+2}) & \ldots & \widetilde{\mu}_1(a_{n-1}) & & \widetilde{\mu}_1(a_n)
\end{array}$$

Corollary 3.3.14. *Given a natural number* $n \in \mathbb{N}^* \setminus \{1\}$ *there is a hypergroup* H *such that* $s.f.g.(H) = n$. *Moreover,* H *is a join space.*

Proof. We consider the hypergroupoid $H = \{x_1, x_2, ..., x_p\}$, $p = 2^n$, with the hyperoperation:

$$x_i \circ x_i = x_i, \quad \text{for} \quad i \in \{1, 2, ..., p\}$$
$$x_i \circ x_j = x_j \circ x_i = \{x_i, x_{i+1}, ..., x_j\}, \quad \text{for} \quad 1 \leq i < j \leq p,$$

that is, H is the join space ${}^1H$ associated with the hypergroupoid of the previous theorem and therefore the join space ${}^nH$ is a total hypergroup and $s.f.g.(H){=}n$.

Theorem 3.3.15. *Let us consider the decomposition* ${}^iH = \bigcup_{l=1}^{r} C_l$*, where, for all* l*,* $1 \leq l \leq r$ *and for all* $x \in C_l$*,* $C_l = \widetilde{\mu}_{i-1}^{-1}(\widetilde{\mu}_{i-1}(x))$ *and let* $(k_1, k_2, ..., k_r)$ *be the* r*-tuple associated with the join space* ${}^iH$*.*
If $(k_1, k_2, ..., k_r) = (k_r, k_{r-1}, ..., k_1)$*, then the join spaces* ${}^{i+1}H$ *and* ${}^iH$ *are not isomorphic.*

Proof. The condition $(k_1, k_2, ..., k_r) = (k_r, k_{r-1}, ..., k_1)$ can be written in the form

(1) $$k_l = k_{r+1-l}, \quad \text{for any l}, \quad 1 \leq l \leq \left[\frac{r}{2}\right].$$

To prove the theorem we shall show that, for any $x \in C_l$ and any $y \in C_{r+1-l}$, $\widetilde{\mu}_i(x) = \widetilde{\mu}_i(y)$. It follows, then, that ${}^{i+1}H = \bigcup_{l=1}^{r'} C'_l$, where $r' = \left[\frac{r+1}{2}\right]$.
This means that the join space ${}^{i+1}H$ is not isomorphic with ${}^iH$.

Let us take two arbitrary elements $x \in C_l$ and $y \in C_{r+1-l}$.
First we prove that

(2) $$q(x) = q(y).$$

By the condition (1) we have $k_l = k_{r+1-l}$ and by the Theorem 3.3.1, we obtain
$q(x) = 2_l k^l k - k_l^2$ and $q(y) =_{r+1-l} k \ {}^{r+1-l}k - k_{r+1-l}^2$
where
${}_lk = k_1 + k_2 + ... + k_l = k_r + k_{r-1} + ... + k_{r+1-l} =^{r+1-l} k$ and
${}^lk = k_l + k_{l+1} + ... + k_r = k_{r+1-l} + k_{r-l} + ... + k_1 =_{r+1-l} k$,
so ${}_lk =^{r+1-l} k$ and ${}^lk =_{r+1-l} k$.
It is clear that $q(x) = q(y)$.

Now we prove that

(3) $$A(x) = A(y).$$

First we remark that, for any j, j', we have
$k_{l+j} = k_{r-(l+j)+1} = k_{r-l-j+1} = k_{(r-l+1)-j}$
and then

(4) $$\sum_{t=j+1}^{(r-l+1)-j'} k_t = \sum_{t=l+j'}^{r-j} k_t$$

because

$$\sum_{t=j+1}^{(r-l+1)-j'} k_t = k_{j+1} + k_{j+2} + ... + k_{(r-l+1)-j'} =$$

$$= k_{r-j} + k_{r-j-1} + ... + k_{r-(r-l+1-j')+1} =$$

$$= k_{r-j} + k_{r-j-1} + ... + k_{l+j'} = \sum_{j+j'}^{r-j} k_t$$

We shall determine the formulas of $A(x)$, $A(y)$, according to the Theorem 3.3.1. One calculates:

$$\begin{aligned} A(x) &= k_l + \sum_{\substack{i\le l\le j\\ i\neq j}}^{i,j} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} = k_l + \sum_{i=1}^{l-1}\sum_{j=l}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} = \\ &= k_l + \sum_{i=1}^{l-1}\sum_{j=l}^{r-l} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + \sum_{i=1}^{l-1}\sum_{j=r-l+1}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + \sum_{j=l+1}^{r-l} \frac{k_l k_j}{\sum\limits_{t=l}^{j} k_t} + \sum_{j=r-l+1}^{r} \frac{k_l k_j}{\sum\limits_{t=l}^{j} k_t} = \\ &= k_l + \sum_{i=1}^{l}\sum_{j=r-l+1}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + \sum_{i=1}^{l-1}\sum_{j=l}^{r-l} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + k_l \sum_{j=l+1}^{r-l} \frac{k_j}{\sum\limits_{t=l}^{j} k_t}. \end{aligned}$$

Similarly we find

$$\begin{aligned} A(y) &= k_{r-l+1} + \sum_{\substack{i\le r-l+1\le j\\ i\neq j}}^{i,j} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} = \\ &= k_{r-l+1} + \sum_{i=1}^{r-l}\sum_{j=r-l+1}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + \sum_{j=r-l+2}^{r} \frac{k_{r-l+1} k_j}{\sum\limits_{t=r-l+1}^{j} k_t} + \sum_{j=l+1}^{r-l} \frac{k_l k_j}{\sum\limits_{t=l}^{j} k_t} = \\ &= k_{r-l+1} + \sum_{i=1}^{l}\sum_{j=r-l+1}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + \sum_{i=l+1}^{r-l}\sum_{j=r-l+1}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + k_{r-l+1} \sum_{j=r-l+2}^{r} \frac{k_j}{\sum\limits_{t=r-l+1}^{j} k_t}. \end{aligned}$$

Whence, since $k_l = k_{r-l+1}$, we conclude that $A(x) = A(y)$ if and only if

$$\sum_{i=1}^{l-1}\sum_{j=l}^{r-l} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + k_l \sum_{j=l+1}^{r-l} \frac{k_j}{\sum\limits_{t=l}^{j} k_t} = \sum_{i=l+1}^{r-l}\sum_{j=r-l+1}^{r} \frac{k_i k_j}{\sum\limits_{i\le t\le j} k_t} + k_l \sum_{j=r-l+2}^{r} \frac{k_j}{\sum\limits_{t=r-l+1}^{j} k_t}$$

if and only if

$$k_1\left(\frac{k_l}{\sum\limits_{t=1}^{l}k_t}+\frac{k_{l+1}}{\sum\limits_{t=1}^{l+1}k_t}+\ldots+\frac{k_{r-l}}{\sum\limits_{t=1}^{r-l}k_t}\right)+k_2\left(\frac{k_l}{\sum\limits_{t=2}^{l}k_t}+\frac{k_{l+1}}{\sum\limits_{t=2}^{l+1}k_t}+\ldots+\frac{k_{r-l}}{\sum\limits_{t=2}^{r-l}k_t}\right)+\ldots+$$

$$+k_{l-1}\left(\frac{k_l}{\sum\limits_{t=l-1}^{l}k_t}+\frac{k_{l+1}}{\sum\limits_{t=l-1}^{l+1}k_t}+\ldots+\frac{k_{r-l}}{\sum\limits_{t=l-1}^{r-l}k_t}\right)+k_l\left(\frac{k_{l+1}}{\sum\limits_{t=l}^{l+1}k_t}+\frac{k_{l+2}}{\sum\limits_{t=l}^{l+2}k_t}+\ldots+\frac{k_{r-l}}{\sum\limits_{t=l}^{r-l}k_t}\right)=$$

$$=k_r\left(\frac{k_{l+1}}{\sum\limits_{t=l+1}^{r}k_t}+\frac{k_{l+2}}{\sum\limits_{t=l+2}^{r}k_t}+\ldots+\frac{k_{r-l+1}}{\sum\limits_{t=r-l+1}^{r}k_t}\right)+k_{r-1}\left(\frac{k_{l+1}}{\sum\limits_{t=l+1}^{r-1}k_t}+\ldots+\frac{k_{r-l+1}}{\sum\limits_{t=r-l+1}^{r-1}k_t}\right)+\ldots+$$

$$+k_{r-l+2}\left(\frac{k_{l+1}}{\sum\limits_{t=l+1}^{r-l+2}k_t}+\ldots+\frac{k_{r-l+1}}{\sum\limits_{t=r-l+1}^{r-l+2}k_t}\right)+k_{r-l+1}\left(\frac{k_{l+1}}{\sum\limits_{t=l+1}^{r-l+1}k_t}+\ldots+\frac{k_{r-l}}{\sum\limits_{t=r-l}^{r-l+1}k_t}\right).$$

By the relation (4), the two terms of the identity are equal and therefore $A(x) = A(y)$. Now the proof is complete.

Remark. The condition expressed in the Theorem 3.3.15 is only a sufficient condition and not a necessary one, as we can see from the following example.

Example 3.3.16. Let us consider the hypergroupoid $H = \{0, 1, 2, 3, 4, 5\}$ endowed with the fuzzy set $\widetilde{\mu}$ which verifies the relation

$$\begin{array}{ccccccc} \widetilde{\mu}(0) & < & \widetilde{\mu}(2) & < & \widetilde{\mu}(3) & < & \widetilde{\mu}(5). \\ \| & & & & \| & & \\ \widetilde{\mu}(1) & & & & \widetilde{\mu}(4) & & \end{array}$$

Clearly, the 4-tuple corresponding to H is $(2, 1, 2, 1)$, which does not verify the condition of the Theorem 3.3.15. Using the usual formula for calculating the membership function $\widetilde{\mu}_1$ one obtains:

$$\begin{array}{lll} A(0) = 2 + \frac{4}{3} + \frac{8}{5} + \frac{4}{6} = \frac{28}{5} & ; \ q(0) = 20 & ; \ \widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,28 \\ A(2) = 1 + \frac{8}{3} + \frac{8}{5} + \frac{4}{6} + \frac{2}{4} = \frac{193}{30} & ; \ q(2) = 23 & ; \ \widetilde{\mu}_1(2) = 0,279 \\ A(3) = 2 + \frac{8}{5} + \frac{4}{6} + \frac{8}{3} + \frac{1}{2} = \frac{223}{30} & ; \ q(3) = 26 & ; \ \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,285 \\ A(5) = 1 + \frac{4}{6} + \frac{1}{2} + \frac{4}{3} = \frac{7}{2} & ; \ q(5) = 11 & ; \ \widetilde{\mu}_1(5) = 0,318 \end{array}$$

In consequence we have

$$\begin{array}{ccccccc}
\widetilde{\mu}_1(2) & < & \widetilde{\mu}_1(0) & < & \widetilde{\mu}_1(3) & < & \widetilde{\mu}_1(5) \\
 & & \| & & \| & & \\
 & & \widetilde{\mu}_1(1) & & \widetilde{\mu}_1(4) & &
\end{array}$$

and therefore its associated 4-tuple is $(1, 2, 2, 1)$ and thus the join space ${}^2H$ is not isomorphic with ${}^1H$.

3.3.4 Fuzzy grade of the complete hypergroups

Any complete hypergroup H can be represented as $H = \bigcup_{g\in G} A_g$, where

1) $(G, \cdot)$ is a group;

2) for any $(g_1, g_2) \in G^2, g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$;

3) if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.

If G is a commutative group, then H is a complete commutative hypergroup, so a join space.

By the above representation, any complete hypergroup of order n is characterized by a m-tuple $[k_1, k_2, ..., k_m]$, where $m = |G|$, $2 \leq m \leq n-1$, $G = \{g_1, g_2, ..., g_m\}$ and, for any $i \in \{1, 2, ..., m\}$, $k_i = \mid A_{g_i} \mid$.

Theorem 3.3.17. *Let H be a complete hypergroup. Then for any $u \in H$, we have $\widetilde{\mu}(u) = \dfrac{1}{\mid A_{g_u} \mid}$, where $u \in A_{g_u}$.*

Proof. By hypothesis, $\forall u \in H$, there is a unique $g_u \in G$, so that $u \in A_{g_u}$. From (ω) we obtain

$$\forall u \in H,\ A(u) = \sum_{\substack{(x,y)\in H^2 \\ u\in x\circ y}} \mu_{xy}(u) = \sum_{\substack{(x,y)\in H^2 \\ u\in x\circ y}} \frac{1}{|x\circ y|} = \sum_{\substack{(x,y)\in H^2 \\ u\in x\circ y}} \frac{1}{|A_{g_x g_y}|} =$$

$$= \sum_{\substack{(x,y)\in H^2 \\ u\in x\circ y}} \frac{1}{|A_{g_u}|} = \frac{q(u)}{|A_{g_u}|}.$$

So, $\widetilde{\mu}(u) = \dfrac{1}{|A_{g_u}|}$.

We used that $u \in x \circ y = A_{g_x g_y}$ and g_u is unique in G with the property $u \in A_{g_u}$ imply that $g_x g_y = g_u$.

Definition 3.3.18. Let n be a positive integer, $n \geq 3$. A *k-decomposition* of n, $2 \leq k \leq n-1$, is an ordered system of natural numbers $(m_1, m_2, ..., m_k)$ such that $m_i \geq 1$, for any i, $1 \leq i \leq k$, $m_1 + m_2 + ... + m_k = n$, $m_2 \leq m_3 \leq ... \leq m_k$.

We denote $d_k(n)$ the number of k-decompositions of n.

Theorem 3.3.19. *There are* $\sum_{k=2}^{n-1} s_k d_k(n)$ *non-isomorphic complete hypergroups of order* n*, where* s_k *denotes the number of the non-isomorphic groups of order* k*.*

Proof. We fix an arbitrary group G of order k and let H be a complete hypergroup of order n. Since $H = A_{g_1} \cup A_{g_2} \cup ... \cup A_{g_k}$, we obtain $|H| = \sum_{i=1}^{k} |A_{g_i}|$. Hence there are $d_k(n)$ possibilities of making a partition of H as in Theorem 1.17.

It follows that the class of complete hypergroups of order n has the cardinality $\sum_{k=2}^{n-1} s_k d_k(n)$.

Remark. The fuzzy grade of a complete hypergroup does not depend on the considered group G, but only on the k-decompositions of n.

Theorem 3.3.20. (see [19]) *Let* H *be a complete hypergroup of order* $n \leq 6$.

(i) *For* $n = 3$, $s.f.g.(H) = 1$.

(ii) *For* $n = 4$, *there are* 3 *hypergroups with* $s.f.g.(H) = 1$ *and two hypergroups with* $s.f.g.(H) = 2$.

(iii) *For* $n = 5$, $s.f.g.(H) = 1$.

(iv) *For* $n = 6$, *there are* 17 *hypergroups of* $s.f.g.(H) = 1$ *and* 4 *hypergroups of* $s.f.g.(H) = 2$.

The proof is detailed in the Appendix A.

Next we determine the fuzzy grades of some complete 1-hypergroups of a given structure.

If H is a complete 1-hypergroup, then it is of the type $[1, k_2, k_3, ..., k_m]$.

Proposition 3.3.21. *If* H *is a complete* 1*-hypergroup of the type* $[\underbrace{1, 1, ..., 1}_{k \quad times}, k]$*, where* $n = |H| = 2k$*, then* $s.f.g.(H)$*=2.*
More general, if the structure of the complete hypergroup H*, which is not an* 1*-hypergroup, is represented by* $[\underbrace{p, p, ..., p}_{k \quad times}, kp]$*, where* $n = |H| = 2kp$*, then* $s.f.g.(H)$*=2.*

Proof. We suppose that the structure of H is determined by the $(k + 1)$-tuple $[\underbrace{p, p, ..., p}_{k \quad times}, kp]$. By the Theorem 3.3.17, we obtain that the pair associated with the hypergroup $(H, \widetilde{\mu})$ is (kp, kp), therefore we conclude that the join space ${}^2H$ associated with H is a total hypergroup and $s.f.g.(H) = 2$.

Proposition 3.3.22. *Let H be a complete 1-hypergroup of the following type:* $[\underbrace{1,1,...,1}_{l\ \ times},\underbrace{k,k,...,k}_{p\ \ times},l]$, *with* $1 < k < l$, $p \geq 1$, $n = \mid H \mid = pk + 2l$.

(i) *If* $kp = 2l$ ($\Longleftrightarrow n = 4l$), *then* $s.f.g.(H) = 3$.

(ii) *If* $kp \neq 2l$, *then* $f.g.(H) = 2$.

Proof. We use again the Theorem 3.3.17.

(i) If $kp = 2l$, then, with the hypergroupoid $(H, \widetilde{\mu})$ is associated the triple $(l, 2l, l)$ and, by the Case II of the section 3.3.3, with the join space ${}^1H$ is associated the pair $(2l, 2l)$ and then, by the previous Case I, with the join space ${}^2H$ is associated $(4l)$; thus ${}^3H$ is a total hypergroup and $s.f.g.(H) = 3$.

(ii) If $kp \neq 2l$, then the triple associated with the hypergroup $(H, \widetilde{\mu})$ is (l, kp, l); with the join space ${}^1H$ it is associated the pair $(2l, kp)$ and by consequence $f.g.(H) = 2$.

Remark 3.3.23. Generalizing the Proposition 3.3.22 for the complete hypergroups which are not 1-hypergroups, but are of the type $[\underbrace{p,p,...,p}_{s\ \ times},\underbrace{k,k,...,k}_{t\ \ times},ps]$, with $2 \leq p < k < ps$, $n = |H| = 2ps + kt$, we obtain that for $n = 4ps$, $s.f.g.(H) = 3$ and for $kt \neq 2ps$, $f.g.(H) = 2$.

Proposition 3.3.24. *Let H be a complete 1-hypergroup of the following type:* $[\underbrace{1,1,...,1}_{l\ \ times},\underbrace{k,k,...,k}_{p\ \ times},\underbrace{p,p,...,p}_{k\ \ times},l]$, *with* $1 < k < p < l$, $n = \mid H \mid = 2(l + pk)$.

(i) *If* $pk = l$ ($\Longleftrightarrow n = 4l$), *then* $s.f.g.(H) = 3$.

(ii) *If* $pk \neq l$, *then* $f.g.(H) = 2$.

Proof. We use again the Theorem 3.3.17. With the hypergroupoid $(H, \widetilde{\mu})$ is associated the 4-tuple (l, pk, pk, l) and then, with the join space ${}^1H$ is associated the pair $(2l, 2pk)$.

If $l = pk$ then with the join space ${}^2H$ is associated $(4l)$; thus ${}^3H$ is a total hypergroup and $s.f.g.(H) = 3$.

If $pk \neq l$, by the Case I of the section 3.3.3, we have $f.g.(H) = 2$.

Remark 3.3.25. Generalizing the Proposition 3.3.24 for the complete hypergroups of the type $[\underbrace{k,k,...,k}_{l\ \ times},\underbrace{p,p,...,p}_{s\ \ times},\underbrace{s,s,...,s}_{p\ \ times},\underbrace{l,l,...,l}_{k\ \ times}]$, (which are not 1-hypergroups), with $2 \leq k < p < s < l$, $n = |H| = 2(ps + kl)$, we obtain that, for $kl = ps$, $s.f.g.(H) = 3$ and for $kl \neq ps$, $f.g.(H) = 2$.

3.3.5 An example of a non-complete 1-hypergroup

We present an example of a non-complete 1-hypergroup and we study the sequence of join spaces and fuzzy sets associated with it.

We consider the following hypergroup $H = H_n = \{e\} \cup A \cup B$, where $\mid A \mid = \alpha$, $\mid B \mid = \beta$ with $\alpha, \beta \geq 2$ and $A \cap B = \emptyset$, $e \notin A \cup B$. We set $A = \{a_1, ..., a_\alpha\}$ and $B = \{b_1, ..., b_\beta\}$. The hyperoperation is defined in this way:

$$\forall\, a \in A,\ a \circ a = b_1,$$

$$\forall\, (a_1, a_2) \in A^2 \quad \text{such that} \quad a_1 \neq a_2,\ a_1 \circ a_2 = B,$$

$$\forall\, (a, b) \in A \times B,\ a \circ b = b \circ a = e,$$

$$\forall\, (b, b') \in B^2,\ b \circ b' = A,$$

$$\forall\, a \in A,\ a \circ e = e \circ a = A,$$

$$\forall\, b \in B,\ b \circ e = e \circ b = B \text{ and } e \circ e = e.$$

H_n is an 1-hypergroup which is not complete.

§1. Let us suppose $n = |H_6| = 6$, where $H = H_6 = \{e\} \cup A \cup B$, $\alpha = |A| = 2$, $\beta = |B| = 3$, $A \cap B = \emptyset$, $e \notin A \cup B$ with $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$. So the structure in H_6 is as follows:

H	e	a_1	a_2	b_1	b_2	b_3
e	e	A	A	B	B	B
a_1		b_1	B	e	e	e
a_2			b_1	e	e	e
b_1				A	A	A
b_2					A	A
b_3						A

therefore we have: $\widetilde{\mu}(e)$=1; $\widetilde{\mu}(a_1)$=$\widetilde{\mu}(a_2)$=0, 5; $\widetilde{\mu}(b_1)$=0, 467; $\widetilde{\mu}(b_2)$=$\widetilde{\mu}(b_3)$=0, 333, whence the join space ${}^1H$ associated is

${}^1H$	e	a_1	a_2	b_1	b_2	b_3
e	e	$A \cup \{e\}$	$A \cup \{e\}$	$H \setminus \{b_1, b_2\}$	H	H
a_1		A	A	$A \cup \{b_1\}$	$H \setminus \{e\}$	$H \setminus \{e\}$
a_2			A	$A \cup \{b_1\}$	$H \setminus \{e\}$	$H \setminus \{e\}$
b_1				b_1	B	B
b_2					b_2, b_3	b_2, b_3
b_3						b_2, b_3

Now we obtain: $\widetilde{\mu}_1(e) = 0, 32$; $\widetilde{\mu}_1(a_i) = 0, 286$; $\widetilde{\mu}_1(b_1) = 0, 280 \approx 0, 2797$; $\widetilde{\mu}_1(b_2) = \widetilde{\mu}_1(b_3) = 0, 280$ so $\widetilde{\mu}_1(b_1) < \widetilde{\mu}_1(b_j) < \widetilde{\mu}_1(a_i) < \widetilde{\mu}_1(e)$. Therefore the join space ${}^2H$

associated is the following one

^2H	e	a_1	a_2	b_1	b_2	b_3
e	e	$A\cup\{e\}$	$A\cup\{e\}$	H	$H\setminus\{b_1\}$	$H\setminus\{b_1\}$
a_1		A	A	$H\setminus\{e\}$	$H\setminus\{e,b_1\}$	$H\setminus\{e,b_1\}$
a_2			A	$H\setminus\{e\}$	$H\setminus\{e,b_1\}$	$H\setminus\{e,b_1\}$
b_1				b_1	B	B
b_2					b_2,b_3	b_2,b_3
b_3						b_2,b_3

therefore $\widetilde{\mu}_2(e)=\widetilde{\mu}_2(b_1)=0,315$; $\widetilde{\mu}_2(a_1)=\widetilde{\mu}_2(a_2)=\widetilde{\mu}_2(b_2)=\widetilde{\mu}_2(b_3)=0,279$. It follows that the join space ^3H associated is:

^3H	e	a_1	a_2	b_1	b_2	b_3
e	e,b_1	H	H	e,b_1	H	H
a_1		$H\setminus\{e,b_1\}$	$H\setminus\{e,b_1\}$	H	$H\setminus\{e,b_1\}$	$H\setminus\{e,b_1\}$
a_2			$H\setminus\{e,b_1\}$	H	$H\setminus\{e,b_1\}$	$H\setminus\{e,b_1\}$
b_1				e,b_1	H	H
b_2					$H\setminus\{e,b_1\}$	$H\setminus\{e,b_1\}$
b_3						$H\setminus\{e,b_1\}$

Now we obtain: $\widetilde{\mu}_3(e)=\widetilde{\mu}_3(b_1)=0,246$; $\widetilde{\mu}_3(a_i)=\widetilde{\mu}_3(b_2)=\widetilde{\mu}_3(b_3)=0,208$, therefore $\forall\, r\geq 3$, we have $^rH={}^3H$ and $s.f.g.(H)=3$.

§2. More generally, set $H=H_n=\{e\}\cup A\cup B$, $|A|=\alpha$, $|B|=\beta$, $\alpha,\beta\geq 2$, $A\cap B=\emptyset$, $e\notin A\cup B$, where $A=\{a_1,...,a_\alpha\}$, $B=\{b_1,...,b_\beta\}$.

First we consider $\alpha=\beta\geq 2$ and therefore the hypergroup H is the following one:

H	e	a_1	a_2	$\cdots$	a_α	b_1	b_2	$\cdots$	b_α
e	e	A	A	$\cdots$	A	B	B	$\cdots$	B
a_1		b_1	B	$\cdots$	B	e	e	$\cdots$	e
a_2			b_1	$\cdots$	B	e	e	$\cdots$	e
$\vdots$				$\ddots$					
a_α					b_1	e	e	$\cdots$	e
b_1						A	A	$\cdots$	A
b_2							A	$\cdots$	A
$\vdots$								$\ddots$	
b_α									A

whence $\widetilde{\mu}(e)=1$; $A(a_i)=\dfrac{2\alpha+\beta+2(1+2+...+\beta-1)}{\alpha}=\dfrac{2\alpha+\beta^2}{\alpha}$; $q(a_i)=2\alpha+\beta^2$, so $\widetilde{\mu}(a_i)=\dfrac{1}{\alpha}\cdot$ In the same way we find for $i\neq 1$, $\widetilde{\mu}(b_i)=\dfrac{1}{\alpha}\cdot$ Moreover, we find $A(b_1)=2\alpha+1$, $q(b_1)=\alpha^2+2\alpha$, whence $\widetilde{\mu}(b_1)=\dfrac{2\alpha+1}{\alpha^2+2\alpha}$, therefore we have $\forall\, i\in I(\alpha)=\{1,...,\alpha\}$ and $\forall\, j\in I(\alpha)-\{1\}$,

$$\widetilde{\mu}(e)>\widetilde{\mu}(b_1)>\widetilde{\mu}(a_i)=\widetilde{\mu}(b_j).$$

Set $B_1 = B \setminus \{b_1\}$, $H^* = H \setminus \{e\}$.
The join space ${}^1H$ associated is

${}^1H$	e	b_1	a_1	a_2	$\cdots$	a_α	b_2	$\cdots$	b_α
e	e	e, b_1	H	H	$\cdots$	H	H	$\cdots$	H
b_1		b_1	H^*	H^*	$\cdots$	H^*	H^*	$\cdots$	H^*
a_1			$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$
a_2				$A \cup B_1$	$\cdots$	$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$
$\vdots$					$\ddots$				
a_α						$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$
b_2							$A \cup B_1$	$\cdots$	$A \cup B_1$
$\vdots$								$\ddots$	
b_α									$A \cup B_1$

Therefore we have: $A(e) = \dfrac{4n-4}{n}$; $q(e) = 2n-1$, whence $\widetilde{\mu}_1(e) = \dfrac{4(n-1)}{n(2n-1)}$.
$A(b_1) = 2 + \dfrac{2\alpha + 2(\alpha-1)}{n} + \dfrac{2\alpha + 2(\alpha-1)}{n-1} = 2\,\dfrac{3n^2-6n+2}{n(n-1)}$; $q(b_1) = 4n-5$, so $\widetilde{\mu}_1(b_1) = \dfrac{2(3n^2-6n+2)}{n(4n^2-9n+5)}$. We have clearly:

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(b_1) \Longleftrightarrow \frac{2n-2}{2n-1} > \frac{3n^2-6n+2}{4n^2-9n+5} \Longleftrightarrow 8n^3 - 18n^2 + 10n - 8n^2 + 18n - 10 >$$
$$> 6n^3 - 12n^2 + 4n - 3n^2 + 6n - 2 \Longleftrightarrow G = 2n^3 - 11n^2 + 18n - 8 > 0.$$

We obtain that for $n \geq 5$, $G = n^2(2n-11) + 2(9n-4) > 0$. On the other side $A(a_i) = \dfrac{n^3+n^2-8n+4}{n(n-1)}$; $q(a_i) = (n-2)(n+2)$, whence $\widetilde{\mu}_1(a_i) = \dfrac{n^2+3n-2}{n(n-1)(n+2)}$.
We have also $\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i) \Longleftrightarrow \dfrac{2(3n^2-6n+2)}{4n-5} > \dfrac{n^2+3n-2}{n+2} \Longleftrightarrow F = 2n^3 - 7n^2 + 3n - 2 > 0$, and since $F = n^2(2n-7) + (3n-2)$, we have finally, for every $n \geq 5$, $\forall\, j \in I(\alpha) \setminus \{1\}$ and $\forall\, i \in I(\alpha)$, $\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i) = \widetilde{\mu}_1(b_j)$. Then

$$\forall\, r \geq 1, {}^rH = {}^1H \quad \text{and} \quad s.f.g.(H) = 1.$$

§3. Set now $\alpha > \beta \geq 2$, then we have $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_i) = \dfrac{1}{\alpha}$, $\widetilde{\mu}(b_j) = \dfrac{1}{\beta}$, for $j \in I(\beta) - \{1\}$, so $\widetilde{\mu}(a_i) < \widetilde{\mu}(b_j) < \widetilde{\mu}(e)$. On the other side $\widetilde{\mu}(b_1) = \dfrac{\alpha^2 + \alpha\beta + 2\beta - \alpha}{\beta(\alpha^2 + 2\beta)}$ from which $\widetilde{\mu}(b_1) > \widetilde{\mu}(b_j) \Longleftrightarrow \dfrac{\alpha^2 + \alpha\beta + 2\beta - \alpha}{\beta(\alpha^2 + 2\beta)} > \dfrac{1}{\beta}$, true.

Hence $\widetilde{\mu}(a_i) < \widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) < \widetilde{\mu}(e)$. By consequence, the associated join space is

${}^1H$	e	b_1	b_2	$\cdots$	b_β	a_1	$\cdots$	a_α
e	e	e, b_1	$H \setminus A$	$\cdots$	$H \setminus A$	H	$\cdots$	H
b_1		b_1	B	$\cdots$	B	H^*	$\cdots$	H^*
b_2			B_1	$\cdots$	B_1	$H \setminus \{e, b_1\}$	$\cdots$	$H \setminus \{e, b_1\}$
$\vdots$				$\ddots$				
b_β					B_1	$H \setminus \{e, b_1\}$	$\cdots$	$H \setminus \{e, b_1\}$
a_1						A	$\cdots$	A
$\vdots$							$\ddots$	
a_α								A

where $B_1 = B \setminus \{b_1\}$. Then $A(e) = 2 + \dfrac{2(\beta-1)}{n-\alpha} + \dfrac{2\alpha}{n} = \dfrac{6n\beta+2n-2\beta^2-4\beta-2}{n(\beta+1)}$, and $q(e) = 2n-1$, it follows $\widetilde{\mu}_1(e) = \dfrac{6n\beta+2n-2\beta^2-4\beta-2}{n(2n-1)(\beta+1)}\cdot$ We have also

$$\begin{aligned} A(b_1) &= 2 + \frac{2(\beta-1)}{n-\alpha} + \frac{2\alpha}{n} + \frac{2(\beta-1)}{\beta} + \frac{2\alpha}{n-1} = \\ &= 2\,\frac{5n^2\beta^2 - 8n\beta^2 - 2n\beta^3 + \beta^3 + 2n^2\beta - 3n\beta + 2\beta^2 + \beta - n^2 + n}{n\beta(n-1)(\beta+1)}; \\ q(b_1) &= 1 + 2\beta + 4\alpha + 2\beta - 2 = 4n - 5. \end{aligned}$$

By consequence, $\widetilde{\mu}(e) > \widetilde{\mu}(b_1)$ if and only if $\dfrac{6n\beta+2n-2\beta^2-4\beta-2}{n(2n-1)(\beta+1)} >$ $> 2\,\dfrac{5n^2\beta^2 - 8n\beta^2 - 2n\beta^3 + \beta^3 + 2n^2\beta - 3n\beta + 2\beta^2 + \beta - n^2 + n}{n\beta(n-1)(\beta+1)(4n-5)}$.

Set $P = \beta(n-1)(4n-5)(3n\beta+n-\beta^2-2\beta-1)$, $Q = (2n-1)(5n^2\beta^2 - 8n\beta^2 - 2n\beta^3 + \beta^3 + 2n^2\beta - 3n\beta + 2\beta^2 + \beta - n^2 + n)$. We have

$$\begin{aligned} P > Q \Longleftrightarrow P &= n^3(12\beta^2+4\beta) + n^2(-4\beta^3 - 35\beta^2 - 13\beta) + \\ &+ n(9\beta^3 + 33\beta^2 + 14\beta) - 5\beta^3 - 10\beta^2 - 5\beta > \\ &> Q = n^3(10\beta + 4\beta - 2) + n^2(-4\beta^3 - 21\beta^2 - 8\beta + 3) + \\ &+ n(4\beta^2 + 12\beta^2 + 5\beta - 1) - \beta^3 - 2\beta^2 - \beta. \end{aligned}$$

This is true if and only if $P - Q = n^3(2\beta^2+2) + n^2(-14\beta^2 - 5\beta - 3) + n(5\beta^3 + 21\beta^2 + 9\beta + 1) - 4\beta^3 - 8\beta^2 - 4\beta > 0$. We have clearly $n > 2\beta + 1$.

Let be $P - Q = E_n$, so we have

$$\begin{aligned} E''_n &= 6n(2\beta^2+2) - 28\beta^2 - 10\beta - 6 > 6(2\beta+1)(2\beta^2+2) - \\ &- 28\beta^2 - 10\beta - 6 = 24\beta^3 - 16\beta^2 + 14\beta + 6 > 0, \end{aligned}$$

whence $\forall\, \beta \geq 2$. E'_n is a strictly increasing function, that is $\forall\, n : n > 2\beta + 1$

$$E'_n > E'_{2\beta+1}$$

We obtain for $\forall\, \beta : \beta \geq 2$

$$E'_n > 3(2\beta+1)^2(2\beta^2+2) + 2(2\beta+1)(-14\beta^2-5\beta-3) + 5\beta^3 + \\ +21\beta^2+9\beta+1 = 24\beta^4-27\beta^3+3\beta^2+11\beta+1 > 0,$$

therefore E_n is a strictly increasing function. So $\forall\, n : n > 2\beta+1$ we have

$$E_n > E_{2\beta+1}.$$

Finally, we find $\forall\, \beta \geq 2$

$$P-Q > (8\beta^3+12\beta^3+6\beta+1)(2\beta^2+2)(\,4\beta^2+4\beta+1)(-14\beta^2-5\beta-3)+ \\ +(2\beta+1)(5\beta^3+21\beta^2+9\beta+1)-4\beta^3-8\beta^2-4\beta$$

whence $P-Q > 16\beta^5-22\beta^4-5\beta^3+11\beta^2+2\beta > 0$, then

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(b_1).$$

On the other side we have

$$A(b_j) = \frac{2(\beta-1)}{n-\alpha} + \frac{2\alpha}{n} + \frac{2(\beta-1)}{\beta} + \frac{2\alpha}{n-1} + \beta - 1 + \frac{2\alpha(\beta-1)}{n-2} =$$

$$= \frac{\begin{array}{c}2n(n-1)(n-2)(\beta-1)(2\beta+1)+2\beta(\beta+1)(n-2)(n-\beta-1)(2n-1)+ \\ +n(n-1)\beta(\beta+1)(\beta-1)(3n-2\beta-4)\end{array}}{n(n-1)(n-2)\beta(\beta+1)};$$

$$q(b_j) = 4(\beta-1)+4\alpha+(\beta-1)^2+2\alpha(\beta-1) = 2n-\beta^2+2n\beta-2\beta-5.$$

Therefore we have $\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j)$ if and only if

$$\frac{10n^2\beta^2-16n\beta^2-4n\beta^3+2\beta^3+4n^2\beta-6n\beta+4\beta^2+2\beta-2n^2+2n}{n\beta(n-1)(\beta+1)(4n-5)} >$$

$$> \frac{\begin{array}{c}(2n^3-6n^2+4n)(2\beta^2-\beta-1)+(2n\beta^2+2n\beta-4\beta^2-4\beta)(2n^2-3n-2n\beta+\beta+1)+ \\ +(n^2\beta^2+n^2\beta-n\beta^2-n\beta)(3n\beta-2\beta^2-2\beta-3n+4)\end{array}}{n(n-1)(n-2)(\beta+1)(2n-\beta^2+2n\beta-2\beta-5)\beta}$$

this is true if and only if, setting
$P = (2n^2-9n-n\beta^2+2\beta^2+2n^2\beta-6n\beta+4\beta+10)(10n^2\beta^2-16n\beta^2-4n\beta^3+2\beta^3+4n^2\beta-6n\beta+4\beta^2+2\beta-2n^2+2n)$ and
$Q = (4n-5)(8n^3\beta^2-n^3\beta-2n^3-28n^2\beta^2+6n^2+30n\beta^2+6n\beta-4n-11n^2\beta^3+14n\beta^3-4\beta^3-8\beta^2-4\beta-n^2\beta+3n^3\beta^3-2n^2\beta^4+2n\beta^4)$
we have $P > Q$. One finds:
$P = n^4(20\beta^3+28\beta^2+4\beta-4)+n^3(-18\beta^4-104\beta^3-156\beta^2-32\beta+22)+n^2(4\beta^5+64\beta^4+198\beta^3+302\beta^2+78\beta-38)+n(-10\beta^5-64\beta^4-160\beta^3-228\beta^2-70\beta+20)+4\beta^5+16\beta^4+48\beta^3+48\beta^2+20\beta$, and
$Q = n^4(12\beta^3+32\beta^2-4\beta-8)+n^3(-8\beta^4-59\beta^3-152\beta^2+\beta+34)+n^2(18\beta^4+$

$111\beta^3+260\beta^2+29\beta-46)+n(-86\beta^3-10\beta^4-182\beta^2-46\beta+20)+20\beta^3+40\beta^2+20\beta$.
Then one obtains $\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j)$ if and only if

$$\begin{aligned} P-Q = \ & n^4(8\beta^3-4\beta^2+8\beta+4)+n^3(-10\beta^4-45\beta^3-4\beta^2-33\beta-12)+ \\ & +n^2(4\beta^5+46\beta^4+87\beta^3+42\beta^2+49\beta+8)+n(-10\beta^5-54\beta^4- \\ & -74\beta^3-46\beta^2-24\beta)+4\beta^5+16\beta^4+20\beta^3+8\beta^2>0. \end{aligned}$$

Set now $E_n = P - Q$, $\forall\, \beta \geq 2$.

$$\begin{aligned} E_n''' = \ & 24n(8\beta^3-4\beta^2+8\beta+4)-60\beta^4-270\beta^3-24\beta^2-198\beta-72 > \\ > \ & 24(2\beta+1)(8\beta^3-4\beta^2+8\beta+4)-60\beta^4-270\beta^3-24\beta^2-198\beta-72 = \\ = \ & 324\beta^4-174\beta^3+264\beta^2+186\beta+24>0, \end{aligned}$$

whence $E_n'' > E_{2\beta+1}'' = 272\beta^5-316\beta^4+144\beta^3+192\beta^2+44\beta-8>0$, therefore E_n' is a strictly increasing function. It follows

$$E_n' > E_{2\beta+1}' = 152\beta^6-222\beta^5+24\beta^4+137\beta^3+50\beta^2-9\beta-4>0,$$

so $E_n > E_{2\beta+1} = 64\beta^7-108\beta^6-18\beta^5+72\beta^4+40\beta^3-6\beta^2-8\beta>0$, therefore

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j).$$

On the other side we have

$$\begin{aligned} A(a_i) &= \frac{2\alpha}{n}+\frac{2\alpha}{n-1}+\frac{2\alpha(\beta-1)}{n-2}+\alpha = \\ &= \frac{(n-\beta-1)(n^3-n^2-6n+2n^2\beta-2n\beta+4)}{n(n-1)(n-2)} \end{aligned}$$

$$q(a_i) = 4\alpha+\alpha^2+2\alpha(\beta-1) = (n-\beta-1)(n+\beta+1)$$

therefore $\widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(a_i)$ if and only if

$$\frac{\begin{array}{c} 8n^3\beta^2-n^3\beta-2n^3-28n^2\beta^2+6n^2+30n\beta^2+6n\beta-4n-11n^2\beta^3+ \\ +14n\beta^3-4\beta^3-8\beta^2-4\beta-n^2\beta+3n^3\beta^3-2n^2\beta^4+2n\beta^4 \end{array}}{n(n-1)(n-2)\beta(\beta+1)(2n-\beta^2+2n\beta-2\beta-5)} >$$

$$> \frac{n^3-n^2-6n+2n^2\beta-2n\beta+4}{n(n-1)(n-2)(n+\beta+1)}$$

This is true if and only if
$P = (n+\beta+1)(8n^3\beta^2-n^3\beta-2n^3-28n^2\beta^2+6n^2+30n\beta^2+6n\beta-4n-11n^2\beta^3+14n\beta^3-4\beta^3-8\beta^2-4\beta-n^2\beta+3n^3\beta^3-2n^2\beta^4+2n\beta^4) >$
$> Q = (2n\beta^2+2n\beta-\beta^4-\beta^3+2n\beta^3+2n\beta^2-2\beta^3-2\beta^2-5\beta^2-5\beta)(n^3-n^2-6n+2n^2\beta-2n\beta+4)$.
One obtains
$P = n^4(3\beta^3+8\beta^2-\beta-2)+n^3(\beta^4-21\beta^2-4\beta+4)+n^2(-2\beta^5-11\beta^4-25\beta^3+$

$\beta^2+11\beta+2)+n(2\beta^5+16\beta^4+40\beta^3+28\beta^2-2\beta-4)-4\beta^4-12\beta^3-12\beta^2-4\beta$ and
$Q=n^4(2\beta^3+4\beta^2+2\beta)+n^3(3\beta^4+3\beta^3-7\beta^2-7\beta)+n^2(-2\beta^5-9\beta^4-31\beta^3-31\beta^2-7\beta)+n(2\beta^5+12\beta^4+40\beta^3+68\beta^2+38\beta)-4\beta^4-12\beta^3-28\beta^2-20\beta$.
So $P-Q=n^4(\beta^3+4\beta^2-3\beta-2)+n^3(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)+n^2(-2\beta^4+6\beta^3+32\beta^2+18\beta+2)+n(4\beta^4-40\beta^2-40\beta-4)+16\beta^2+16\beta$.

Set $E_n=P-Q$, then we have

$$\begin{aligned}E_n''' &= 24n(\beta^3+4\beta^2-3\beta-2)-12\beta^4-18\beta^3-84\beta^2+18\beta+24\geq\\ &\geq(48\beta+24)(\beta^3+4\beta^2-3\beta-2)-12\beta^4-18\beta^3-84\beta^2+18\beta+24=\\ &=48\beta^4+216\beta^3-48\beta^2-168\beta-48-12\beta^4-18\beta^3-84\beta^2+18\beta+24=\\ &=36\beta^4+198\beta^3-132\beta^2-150\beta-24>0,\end{aligned}$$

whence E_n'' is a strictly increasing function. It follows

$$\begin{aligned}&E_n''>E_{2\beta+1}''=12(4\beta^2+4\beta+1)(\beta^3+4\beta^2-3\beta-2)+6(2\beta+1)\cdot\\ &\cdot(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)-4\beta^4+12\beta^3+64\beta^2+36\beta+4=\\ &=48\beta^5+192\beta^4-144\beta^3-96\beta^2+48\beta^4+192\beta^3-144\beta^2-96\beta+\\ &+12\beta^3+48\beta^2-36\beta-24-24\beta^5-36\beta^4-168\beta^3+36\beta^2+48\beta-\\ &-12\beta^4-18\beta^3-84\beta^2+18\beta+24-4\beta^4+12\beta^3+64\beta^2+36\beta+4=\\ &=24\beta^5+188\beta^4-114\beta^3-176\beta^2-30\beta+4>0,\end{aligned}$$

and, by consequence, E_n' is a strictly increasing function. So we have

$$\begin{aligned}&E_n'>E_{2\beta+1}'=4(8\beta^3+12\beta^2+6\beta+1)(\beta^3+4\beta^2-3\beta-2)+\\ &+3(4\beta^2+4\beta+1)(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)+\\ &+2(2\beta+1)(-2\beta^4+6\beta^3+32\beta^2+18\beta+2)+4\beta^4-40\beta^2-40\beta-4=\\ &=8\beta^6+108\beta^5-66\beta^4-109\beta^3-14\beta^2+\beta+4>0,\end{aligned}$$

therefore E_n is a strictly increasing function and, by consequence,

$$\begin{aligned}&E_n>E_{2\beta+1}=(16\beta^4+32\beta^3+24\beta^2+8\beta+1)(\beta^3+4\beta^2-3\beta-2)+\\ &+(8\beta^3+12\beta^2+6\beta+1)(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)+\\ &+(4\beta^2+4\beta+1)(-2\beta^4+6\beta^3+32\beta^2+18\beta+2)+\\ &+(2\beta+1)(4\beta^4-40\beta^2-40\beta-4)+16\beta^2+16\beta\end{aligned}$$

whence $E_n=P-Q>E_{2\beta+1}=40\beta^6-32\beta^5-34\beta^4+4\beta^3-8\beta^2+2\beta>0$. Finally,

$$\widetilde{\mu}_1(b_j)>\widetilde{\mu}_1(a_i).$$

We conclude that

$$\widetilde{\mu}_1(a_i)<\widetilde{\mu}_1(b_j)<\widetilde{\mu}_1(b_1)<\widetilde{\mu}_1(e).$$

It follows $\forall r : r \geq 1, {}^{r}H = {}^{1}H$ and $s.f.g.(H) = 1$.

§4. Let $\beta > \alpha \geq 2 \Longrightarrow n > 2\alpha + 1$. We find in the same way as in §3:

$$\widetilde{\mu}(e) = 1, \quad \widetilde{\mu}(a_i) = \frac{1}{\alpha}, \quad \text{for } j \in \{2, 3, ..., \beta\},$$

$$\widetilde{\mu}(b_j) = \frac{1}{\beta}, \quad \widetilde{\mu}(b_1) = \frac{\alpha^2 + \alpha\beta + 2\beta - \alpha}{\beta(\alpha^2 + 2\beta)}$$

4.1. Let us consider now the case $\alpha = 2,\ \beta = n - 3$, whence we have:

$$\widetilde{\mu}(b_1) < \widetilde{\mu}(a_i) \text{ iff } \frac{4\beta + 2}{\beta(2\beta + 4)} < \frac{1}{2} \text{ iff } \beta^2 - 2\beta - 2 > 0 \text{ iff } \beta \geq 3.$$

Moreover, for $j \in I(\beta)$,

$$\widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) \text{ iff } \frac{1}{\beta} < \frac{4\beta + 2}{\beta(2\beta + 4)} \text{ iff } 2\beta - 2 > 0, \text{ iff } \forall \beta \geq 3.$$

So, we have:

$$\widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) < \widetilde{\mu}(a_i) < \widetilde{\mu}(e).$$

The associated join space is the following

${}^{1}H$	b_2	$\cdots$	b_β	b_1	a_1	a_2	e
b_2	B_1	$\cdots$	B_1	B	H^*	H^*	H
$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_β			B_1	B	H^*	H^*	H
b_1				b_1	$A \cup \{b_1\}$	$A \cup \{b_1\}$	$H \setminus B_1$
a_1					A	A	$A \cup \{e\}$
a_2						A	$A \cup \{e\}$
e							e

We obtain

$$A(e) = 1 + \frac{4}{3} + \frac{2}{4} + \frac{2(\beta - 1)}{n} = \frac{29n - 48}{6n}; \ q(e) = 2n - 1; \ \widetilde{\mu}_1(e) = \frac{29n - 48}{6n(2n - 1)}$$

$$A(a_i) = 2 + \frac{8}{3} + \frac{2}{4} + \frac{2(\beta - 1)}{n} + \frac{4(\beta - 1)}{n - 1}; \ A(a_i) = \frac{67n^2 - 187n + 48}{6n(n - 1)}; \ q(a_i) = 6n - 10$$

whence

$$\widetilde{\mu}_1(a_i) = \frac{67n^2 - 187n + 48}{6n(n-1)(6n-10)}.$$

Moreover,

$$A(b_1) = 1 + \frac{4}{3} + \frac{1}{2} + \frac{4(\beta - 1)}{n - 1} + \frac{2(\beta - 1)}{n} + \frac{2(\beta - 1)}{\beta} = \frac{65n^3 - 392n^2 + 615n - 144}{6n(n-1)(n-3)}$$

$$q(b_1) = 8n - 25$$

whence

$$\widetilde{\mu}_1(b_1) = \frac{65n^3 - 392n^2 + 615n - 144}{6n(n-1)(n-3)(8n-25)}$$

Moreover, for $j \in I(\beta) - \{1\}$, we have

$$A(b_j) = n - 4 + \frac{2(n-4)}{n-3} + \frac{4(n-4)}{n-1} + \frac{2(n-4)}{n} = (n-4)\,\frac{n^3+4n^2-19n+6}{n(n-1)(n-3)}$$

$$q(b_j) = (n-4)(n+4)$$

whence

$$\widetilde{\mu}_1(b_j) = \frac{n^3 + 4n^2 - 19n + 6}{n(n-1)(n-3)(n+4)}$$

For $\beta = 3$, that is $n = 6$, see §1.

For $\beta > 3$ we have

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i)$$

if and only if

$$\frac{29n - 48}{6n(2n-1)} > \frac{67n^2 - 187n + 48}{6n(n-1)(6n-10)}$$

which is equivalent with $40n^3 - 311n^2 + 775n - 432 > 0$, true for $n \geq 7$. So,

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i).$$

Moreover,

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j) \Longleftrightarrow \frac{65n^3 - 392n^2 + 615n - 144}{6n(n-1)(n-3)(8n-25)} > \frac{n^3 + 4n^2 - 19n + 6}{n(n-1)(n-3)(n+4)};$$

this is equivalent with $17n^4 - 174n^3 + 559n^2 - 822n + 324 > 0$. Clearly, the inequivalence is true for any $n \geq 7$. So, we obtain

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j).$$

Finally,

$$\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1) \Longleftrightarrow \frac{67n^2 - 187n + 48}{6n(n-1)(6n-10)} > \frac{65n^3 - 392n^2 + 615n - 144}{6n(n-1)(n-3)(8n-25)}$$

$\Longleftrightarrow 146n^4 - 1777n^3 + 6962n^2 - 9363n + 2160 > 0$.

In conclusion, for $\alpha = 2$ and $\beta > 3$ we have

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j)$$

therefore

$$\forall\, r \geq 1,\ \ {}^rH = {}^1H,\ \ s.f.g.(H) = 1.$$

For $\alpha = 2$ and $\beta = 3$ we have

$$\forall\, r \geq 1,\ {}^{r}H = {}^{3}H \quad \text{and} \quad s.f.g.(H) = 3.$$

Let us suppose now $\beta > \alpha \geq 3$. One can see that

$$1 = \widetilde{\mu}(e) > \widetilde{\mu}(b_1) > \widetilde{\mu}(b_j) \quad \text{and} \quad \widetilde{\mu}(a_i) > \widetilde{\mu}(b_j).$$

Moreover,

$$\widetilde{\mu}(b_1) > \widetilde{\mu}(a_i) \Longleftrightarrow \frac{\alpha^2 + \alpha\beta + 2\beta - \alpha}{\beta(\alpha^2 + 2\beta)} \geq \frac{1}{\alpha} \Longleftrightarrow \alpha^2(\alpha - 1) \geq 2\beta(\beta - \alpha).$$

By consequence, we study two cases:
when $\widetilde{\mu}(b_1) > \widetilde{\mu}(a_i)$ and when $\widetilde{\mu}(a_i) > \widetilde{\mu}(b_1)$.

4.2. In the case $\widetilde{\mu}(e) > \widetilde{\mu}(b_1) > \widetilde{\mu}(a_i) > \widetilde{\mu}(b_j)$ the join space ${}^{1}H$ associated is

${}^{1}H$	e	b_1	a_1	$\cdots$	a_α	b_2	$\cdots$	b_β
e	e	e, b_1	$H \setminus B_1$	$\cdots$	$H \setminus B_1$	H	$\cdots$	H
b_1		b_1	$H^* \setminus B_1$	$\cdots$	$H^* \setminus B_1$	H^*	$\cdots$	H^*
a_1			A	$\cdots$	A	$H^* \setminus \{b_1\}$	$\cdots$	$H^* \setminus \{b_1\}$
$\vdots$				$\ddots$	$\vdots$	$\vdots$		$\vdots$
a_α					A	$H^* \setminus \{b_1\}$	$\cdots$	$H^* \setminus \{b_1\}$
b_2						B_1	$\cdots$	B_1
$\vdots$							$\ddots$	$\vdots$
b_β								B_1

We have

$$A(e) = 2 + \frac{2\alpha}{\alpha + 2} + \frac{2(n - \alpha - 2)}{n} = \frac{6n\alpha + 8n - 2\alpha^2 - 8\alpha - 8}{n(\alpha + 2)}; \quad q(e) = 2n - 1$$

whence

$$\widetilde{\mu}_1(e) = \frac{6n\alpha + 8n - 2\alpha^2 - 8\alpha - 8}{n(2n - 1)(\alpha + 2)}\cdot$$

Moreover

$$A(b_1) = 2 + \frac{2\alpha}{\alpha + 2} + \frac{2\alpha}{\alpha + 1} + \frac{2(n - \alpha - 2)}{n} + \frac{2(n - \alpha - 2)}{n - 1} =$$
$$= \frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n - 1)(\alpha + 1)(\alpha + 2)}$$

$q(b_1) = 4n - 5$ so

$$\widetilde{\mu}_1(b_1) = \frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n - 1)(\alpha + 1)(\alpha + 2)(4n - 5)}$$

We have also

$$A(a_i) = \frac{2\alpha(n-\alpha-2)}{n-2} + \frac{2(n-\alpha-2)}{n-1} + \frac{2(n-\alpha-2)}{n} + \frac{2\alpha}{\alpha+2} + \frac{2\alpha}{\alpha+1} + \alpha =$$

$$= \frac{\begin{array}{c}3n^3\alpha^3 + 17n^3\alpha^2 + 24n^3\alpha + 8n^3 - 2n^2\alpha^4 - 19n^2\alpha^3 - 73n^2\alpha^2 - 98n^2\alpha - 36n^2 + \\ +2n\alpha^4 + 22n\alpha^3 + 84n\alpha^2 + 116n\alpha + 48n - 4\alpha^3 - 20\alpha^2 - 32\alpha - 16\end{array}}{n(n-1)(n-2)(\alpha+1)(\alpha+2)}$$

$$q(a_i) = \alpha^2 + 4\alpha + 4(n-\alpha-2) + 2\alpha(n-\alpha-2) = 2n\alpha + 4n - \alpha^2 - 4\alpha - 8.$$

Then

$$\begin{aligned} A(b_j) &= \frac{2(\beta-1)}{n} + \frac{2(\beta-1)}{n-1} + \frac{2\alpha(\beta-1)}{n-2} + \beta - 1 = \\ &= (n-\alpha-2)\,\frac{n^3 + n^2 + 2n^2\alpha - 2n\alpha - 8n + 4}{n(n-1)(n-2)} \end{aligned}$$

and

$$q(b_j) = 4(\beta-1) + 2\alpha(\beta-1) + (\beta-1)^2 = (n-\alpha-2)(n+\alpha+2),$$

whence

$$\widetilde{\mu}_1(b_j) = \frac{n^3 + n^2(1+2\alpha) + n(-2\alpha-8) + 4}{n(n+\alpha+2)(n-1)(n-2)}.$$

Therefore we have $\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i)$ if and only if

$$\frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n-1)(\alpha+1)(\alpha+2)(4n-5)} >$$

$$> \frac{\begin{array}{c}3n^3\alpha^3 + 17n^3\alpha^2 + 24n^3\alpha + 8n^3 - 2n^2\alpha^4 - 19n^2\alpha^3 - 73n^2\alpha^2 - 98n^2\alpha - 36n^2 + \\ +2n\alpha^4 + 22n\alpha^3 + 84n\alpha^2 + 116n\alpha + 48n - 4\alpha^3 - 20\alpha^2 - 32\alpha - 16\end{array}}{n(n-1)(n-2)(\alpha+1)(\alpha+2)(2n\alpha + 4n - \alpha^2 - 4\alpha - 8)}$$

and this is true if and only if

$$\begin{aligned} &P = (2n^2\alpha + 4n^2 - n\alpha^2 - 8n\alpha - 16n + 2\alpha^2 + 8\alpha + 16)(10n^2\alpha^2 + 24n^2\alpha + \\ &+12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8) > \\ &> Q = (4n-5)(3n^3\alpha^3 + 17n^3\alpha^2 + 24n^3\alpha + 8n^3 - 2n^2\alpha^4 - 19n^2\alpha^3 - \\ &-73n^2\alpha^2 - 98n^2\alpha - 36n^2 + 2n\alpha^4 + 22n\alpha^3 + 84n\alpha^2 + 116n\alpha + \\ &+48n - 4\alpha^3 - 20\alpha^2 - 32\alpha - 16). \end{aligned}$$

We have
$P = n^4(20\alpha^3 + 88\alpha^2 + 120\alpha + 48) + n^3(-18\alpha^4 - 176\alpha^3 - 576\alpha^2 - 728\alpha - 288) +$
$+ n^2(4\alpha^5 + 84\alpha^4 + 494\alpha^3 + 1320\alpha^2 + 1552\alpha + 608) + n(-10\alpha^5 - 114\alpha^4 - 516\alpha^3 -$
$- 1192\alpha^2 - 1312\alpha - 512) + 4\alpha^5 + 36\alpha^4 + 144\alpha^3 + 304\alpha^2 + 320\alpha + 128$ and

$Q = n^4(12\alpha^3+68\alpha^2+96\alpha+32)+n^3(-8\alpha^4-91\alpha^3-377\alpha^2-512a-18)+n^2(18\alpha^4+$ $+183\alpha^3+701\alpha^2+954\alpha+372)+n(-10\alpha^4-126\alpha^3-500\alpha^2-708\alpha-304)+20\alpha^3+$ $+100\alpha^2+160\alpha+8$, whence

$$P-Q=n^4(8\alpha^3+20\alpha^2+24\alpha+16)+n^3(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+ \\ +n^2(4\alpha^5+66\alpha^4+311\alpha^3+619\alpha^2+598\alpha+236)+n(-10\alpha^5-104\alpha^4-390\alpha^3-692\alpha^2- \\ -604\alpha-208)+4\alpha^5+36\alpha^4+124\alpha^3+204\alpha^2+160\alpha+48.$$

Set $P-Q=E_n$. Then we find $\forall\,\alpha,\ \alpha\geq 3$,

$$E_n'''=324\alpha^4+642\alpha^3+438\alpha^2+48\alpha-240>0;$$

it follows that E_n'' is a strictly increasing function, so $E_n''>E_{2\alpha+1}''$ if and only if

$$E_n''>12(4\alpha^2+4\alpha+1)(8\alpha^3+20\alpha^2+24\alpha+16)+ \\ +6(2\alpha+1)(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+ \\ +8\alpha^5+132\alpha^4+622\alpha^3+1238\alpha^2+1196\alpha+472= \\ =272\alpha^5+396\alpha^4-68\alpha^3-388\alpha^2-292\alpha+40>0.$$

It follows that E_n' is a strictly increasing function, so $\forall\,n>2\alpha+1, E_n'>E_{2\alpha+1}'$ whence,

$$E_n'>4(8\alpha^3+12\alpha^2+6\alpha+1)(8\alpha^3+20\alpha^2+24\alpha+16)+ \\ +3(4\alpha^2+4\alpha+1)(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+ \\ +(-10\alpha^5-104\alpha^4-390\alpha^3-692\alpha^2-604\alpha-208)+ \\ +2(2\alpha+1)(4\alpha^5+66\alpha^4+311\alpha^3+619\alpha^2+598\alpha+236)= \\ =152\alpha^6+146\alpha^5-246\alpha^4-351\alpha^3-75\alpha^2+120\alpha+16>0.$$

We obtain that E_n is a strictly increasing function and therefore, $\forall\,n>2\alpha+1$, $E_n>E_{2\alpha+1}$. One finds

$$E_n=P-Q>(16\alpha^4+32\alpha^3+24\alpha^2+8\alpha+1)(8\alpha^3+20\alpha^2+24\alpha+16)+ \\ +(8\alpha^3+12\alpha^2+6\alpha+1)(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+ \\ +(4\alpha^2+4\alpha+1)(4\alpha^5+66\alpha^4+311\alpha^3+619\alpha^2+598\alpha+236)+ \\ +(2\alpha+1)(-10\alpha^5-104\alpha^4-390\alpha^3-692\alpha^2-604\alpha-208)+ \\ +4\alpha^5+36\alpha^4+124\alpha^3+204\alpha^2+160\alpha+48= \\ =64\alpha^7+36\alpha^6-158\alpha^5-130\alpha^4+82\alpha^3+112\alpha^2-6\alpha-12>0$$

and by consequence

$$\widetilde{\mu}_1(b_1)>\widetilde{\mu}_1(a_i).$$

On the other side we have $\widetilde{\mu}_1(e)>\widetilde{\mu}_1(b_1)$ if and only if

$$\frac{6n\alpha+8n-2\alpha^2-8\alpha-8}{n(2n-1)(\alpha+2)}> \\ >\frac{10n^2\alpha^2+24n^2\alpha+12n^2-4n\alpha^3-28n\alpha^2-50n\alpha-24n+2\alpha^3+10\alpha^2+16\alpha+8}{n(n-1)(\alpha+1)(\alpha+2)(4n-5)}.$$

This is true if and only if $P=(n\alpha+n-\alpha-1)(4n-5)(6n\alpha+8n-2\alpha^2-8\alpha-8)>$ $>Q=(2n-1)(10n^2\alpha^2+24n^2\alpha+12n^2-4n\alpha^3-28n\alpha^2-50n\alpha-24n+2\alpha^3+$ $+10\alpha^2+16\alpha+8)$. We obtain

$$\begin{aligned}P&=n^3(24\alpha^2+56\alpha+32)+n^2(-8\alpha^3-94\alpha^2-190\alpha-104)+\\&+n(18\alpha^3+120\alpha^2+214\alpha+112)-10\alpha^3-50\alpha^2-80\alpha-40\end{aligned}$$

and

$$\begin{aligned}Q&=n^3(20\alpha^2+48\alpha+24)+n^2(-8\alpha^3-66\alpha^2-124\alpha-60)+\\&+n(8\alpha^3+48\alpha^2+82\alpha+40)-2\alpha^3-10\alpha^2-16\alpha-8\end{aligned}$$

whence

$$\begin{aligned}P-Q&=n^3(4\alpha^2+8\alpha+8)+n^2(-28\alpha^2-66\alpha-44)+\\&+n(10\alpha^3+72\alpha^2+132\alpha+72)-8\alpha^3-40\alpha^2-64\alpha-32.\end{aligned}$$

Set $E_n=P-Q$, whence $\forall\,\alpha\geq 3$, we have that

$$\begin{aligned}E_n''&=6n(4\alpha^2+8\alpha+8)-56\alpha^2-132\alpha-88>\\&>(12\alpha+6)(4\alpha^2+8\alpha+8)-56\alpha^2-132\alpha-88=\\&=48\alpha^3+96\alpha^2+96\alpha+24\alpha^2+48\alpha+48-56\alpha^2-132\alpha-88>0,\end{aligned}$$

therefore E_n' is a strictly increasing function; from this, one finds $\forall\,\alpha\geq 3$

$$\begin{aligned}E_n'&>3(4\alpha^2+4\alpha+1)(4\alpha^2+8\alpha+8)+2(2\alpha+1)(-28\alpha^2-66\alpha-44)+\\&+10\alpha^3+72\alpha^2+132\alpha+72=48\alpha^4+42\alpha^3-44\alpha^2-56\alpha+8>0,\end{aligned}$$

so E_n is a strictly increasing function and therefore

$$\begin{aligned}&E_n>E_{2\alpha+1}=(8\alpha^3+12\alpha^2+6\alpha+1)(4\alpha^2+8\alpha+8)+\\&+(4\alpha^2+4\alpha+1)(-28\alpha^2-66\alpha-44)+(2\alpha+1)(10\alpha^3+72\alpha^2+132\alpha+72)-\\&-8\alpha^3-40\alpha^2-64\alpha-32=32\alpha^5+20\alpha^4-46\alpha^3-24\alpha^2+26\alpha+4>0.\end{aligned}$$

Then, $\forall\,i\in I(\alpha)$,

$$\widetilde{\mu}_1(e)>\widetilde{\mu}_1(b_1)>\widetilde{\mu}_1(a_i).$$

Moreover we have $\widetilde{\mu}_1(a_i)>\widetilde{\mu}_1(b_j)$ if and only if

$$\begin{aligned}&\frac{\begin{array}{l}3n^3\alpha^3+17n^3\alpha^2+24n^3\alpha+8n^3-2n^2\alpha^4-19n^2\alpha^3-73n^2\alpha^2-98n^2\alpha-36n^2+\\+2n\alpha^4+22n\alpha^3+84n\alpha^2+116n\alpha+48n-4\alpha^3-20\alpha^2-32\alpha-16\end{array}}{n(n-1)(n-2)(\alpha+1)(\alpha+2)(2n\alpha+4n-\alpha^2-4\alpha-8)}>\\&>\frac{n^3+2n^2\alpha+n^2-2n\alpha-8n+4}{n(n-1)(n-2)(n+\alpha+2)}.\end{aligned}$$

Set

$$\begin{aligned}P = {} & (n+\alpha+2)(3n^3\alpha^3+17n^3\alpha^2+24n^3\alpha+8n^3-2n^2\alpha^4-\\ & -19n^2\alpha^3-73n^2\alpha^2-98n^2\alpha-36n^2+2n\alpha^4+22n\alpha^3+\\ & +84n\alpha^2+116n\alpha+48n-4\alpha^3-20\alpha^2-32\alpha-16) =\\ & = n^4(3\alpha^3+17\alpha^2+24\alpha+8)+n^3(\alpha^4+4\alpha^3-15\alpha^2-42\alpha-20)+\\ & +n^2(-2\alpha^5-21\alpha^4-89\alpha^3-160\alpha^2-116\alpha-24)+n(2\alpha^5+26\alpha^4+\\ & +124\alpha^3+264\alpha^2+248\alpha+80)-4\alpha^4-28\alpha^3-72\alpha^2-80\alpha-32\end{aligned}$$

and

$$\begin{aligned}Q = {} & (n^3+2n^2\alpha+n^2-2n\alpha-8n+4)(\alpha^2+3\alpha+2)(2n\alpha+4n-\alpha^2-4\alpha-8) =\\ & = (n^3+2n^2\alpha+n^2-2n\alpha-8n+4)(2n\alpha^3+10n\alpha^2-\alpha^4-7\alpha^3-22\alpha^2+\\ & +16n\alpha-32\alpha+8n-16) = n^4(2\alpha^3+10\alpha^2+16\alpha+8)+\\ & +n^3(3\alpha^4+15\alpha^3+20\alpha^2-8)+n^2(-2\alpha^5-19\alpha^4-87\alpha^3-198\alpha^2-208\alpha-80)+\\ & +n(2\alpha^5+22\alpha^4+108\alpha^3+280\alpha^2+352\alpha+160)-4\alpha^4-28\alpha^3-88\alpha^2-128\alpha-64\end{aligned}$$

whence

$$\begin{aligned}P-Q = {} & n^4(\alpha^3+7\alpha^2+8\alpha)+n^3(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)+\\ & +n^2(-2\alpha^4-2\alpha^3+38\alpha^2+92\alpha+56)+\\ & +n(4\alpha^4+16\alpha^3-16\alpha^2-104\alpha-80)+16\alpha^2+48\alpha+32.\end{aligned}$$

Set $E_n = P-Q$, so $\forall\,\alpha \geq 3$

$$\begin{aligned}E_n'' = {} & 24n(\alpha^3+7\alpha^2+8\alpha)-12\alpha^4-66\alpha^3-210\alpha^2-252\alpha-72 >\\ & > 48\alpha^4+336\alpha^3+384\alpha^2+24\alpha^3+168\alpha^2+192\alpha-12\alpha^4-66\alpha^3-\\ & -210\alpha^2-252\alpha-72 = 36\alpha^4+294\alpha^3+342\alpha^2-60\alpha-72>0,\end{aligned}$$

we obtain

$$\begin{aligned}E_n'' > E_{2\alpha+1}'' = {} & 12(4\alpha^2+4\alpha+1)(\alpha^3+7\alpha^2+8\alpha)+\\ & +6(2\alpha+1)(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)-\\ & -4\alpha^4-4\alpha^3+76\alpha^2+184\alpha+112 =\\ & = 24\alpha^5+236\alpha^4+242\alpha^3-170\alpha^2-116\alpha+40>0.\end{aligned}$$

It follows

$$\begin{aligned}E_n' > E_{2\alpha+1}' = {} & 4(8\alpha^3+12\alpha^2+6\alpha+1)(\alpha^3+7\alpha^2+8\alpha)+\\ & +3(4\alpha^2+4\alpha+1)(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)+\\ & +2(2\alpha+1)(-2\alpha^4-2\alpha^3-38\alpha^2+92\alpha+56)+\\ & +4\alpha^4+16\alpha^3-16\alpha^2-104\alpha-80 =\\ & = 8\alpha^6+108\alpha^5+50\alpha^4-237\alpha^3-105\alpha^2+66\alpha-4>0.\end{aligned}$$

Therefore E_n is a strictly increasing function. Then

$$\begin{aligned} P - Q > & \,(16\alpha^4 + 32\alpha^3 + 24\alpha^2 + 8\alpha + 1)(\alpha^3 + 7\alpha^2 + 8\alpha) + \\ & + (8\alpha^3 + 12\alpha^2 + 6\alpha + 1)(-2\alpha^4 - 11\alpha^3 - 35\alpha^2 - 42\alpha - 12) + \\ & + (4\alpha^2 + 4\alpha + 1)(-2\alpha^4 - 2\alpha^3 + 38\alpha^2 + 92\alpha + 56) + \\ & + (2\alpha + 1)(4\alpha^4 + 16\alpha^3 - 16\alpha^2 - 104\alpha - 80) + 16\alpha^2 + 48\alpha + 32 = \\ & = 24\alpha^6 - 56\alpha^5 - 214\alpha^4 - 70\alpha^3 + 62\alpha^2 - 6\alpha - 4 > 0, \end{aligned}$$

so we have, for every $\beta > \alpha \geq 3$, for every $i \in I(\alpha)$ and $j \in I(\beta) \setminus \{1\}$,

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j)$$

and therefore, $\forall\, r \geq 1$, ${}^rH = {}^1H$ and again we find $s.f.g.(H) = 1$.

4.3. Set $\beta > \alpha \geq 3$ and

$$\widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) < \widetilde{\mu}(a_i) < \widetilde{\mu}(e).$$

The join space ${}^1H$ associated is

${}^1H$	b_2	$\cdots$	b_β	b_1	a_1	$\cdots$	a_α	e
b_2	B_1	$\cdots$	B_1	B	H^*	$\cdots$	H^*	H
$\vdots$		$\ddots$						
b_β			B_1	B	H^*	$\cdots$	H^*	H
b_1				b_1	$A \cup \{b_1\}$	$\cdots$	$A \cup \{b_1\}$	$H \setminus B_1$
a_1					A	$\cdots$	A	$A \cup \{e\}$
$\vdots$						$\ddots$		
a_α							A	$A \cup \{e\}$
e								e

where $B_1 = B \setminus \{b_1\}$, $H^* = H \setminus \{e\}$, $\alpha + \beta + 1 = n$. We have

$$\begin{aligned} A(e) &= 1 + \frac{2\alpha}{\alpha+1} + \frac{2}{n-\beta+1} + \frac{2(\beta-1)}{n} = \\ &= \frac{5\alpha^2 n - 2\alpha^3 - 10\alpha^2 + 15n\alpha - 16\alpha + 8n - 8}{n(\alpha+1)(\alpha+2)} \end{aligned}$$

$q(e) = 2n - 1$, therefore

$$\widetilde{\mu}_1(e) = \frac{5\alpha^2 n - 2\alpha^3 - 10\alpha^2 + 15n\alpha - 16\alpha + 8n - 8}{n(2n-1)(\alpha+1)(\alpha+2)}.$$

Moreover

$$A(a_i) = \frac{2\alpha(\beta-1)}{n-1} + \frac{2(\beta-1)}{n} + \frac{4\alpha}{\alpha+1} + \frac{2}{\alpha+2} + \alpha.$$

One obtains

$$A(a_i) = \frac{\begin{matrix}3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - \\ -35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8\end{matrix}}{n(n-1)(\alpha+1)(\alpha+2)}.$$

We have also $q(a_i) = 2\alpha(\beta-1)+2(\beta-1)+4\alpha+\alpha^2+2 = 2n\alpha+2n-\alpha^2-2\alpha-2$, therefore

$$\widetilde{\mu}_1(a_i) = \frac{A(a_i)}{2n\alpha + 2n - \alpha^2 - 2\alpha - 2}.$$

Then $\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i)$ if and only if

$$\frac{5n\alpha^2 + 15n\alpha + 8n - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8}{n(2n-1)(\alpha+1)(\alpha+2)} >$$

$$> \frac{\begin{matrix}3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - 35n\alpha^2 - \\ -42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8\end{matrix}}{n(n-1)(\alpha+1)(\alpha+2)(2n\alpha + 2n - \alpha^2 - 2\alpha - 2)}.$$

This is true if and only if

$$\begin{aligned} P = {} & (5n\alpha^2 + 15n\alpha + 8n - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8)\cdot \\ & \cdot(2n^2\alpha - 2n\alpha + 2n^2 - 2n - n\alpha^2 + \alpha^2 - 2n\alpha + 2\alpha - 2n + 2) > \\ & > Q = (2n-1)(3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - \\ & - 35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8). \end{aligned}$$

We obtain
$P = n^3(10\alpha^3+40\alpha^2+46\alpha+16)+n^2(-9\alpha^4-59\alpha^3-140\alpha^2-140\alpha-48)+n(2\alpha^5+$
$+ 23\alpha^4 + 89\alpha^3 + 160\alpha^2 + 142\alpha + 48) - 2\alpha^5 - 14\alpha^4 - 40\alpha^3 - 60\alpha^2 - 48\alpha - 16$ and
$Q = n^3(6\alpha^3+30\alpha^2+44\alpha+12)+n^2(-4\alpha^4-29\alpha^3-85\alpha^2-106\alpha-34)+n(2\alpha^4+$
$+ 17\alpha^3 + 55\alpha^2 + 74\alpha + 30) - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8.$

Then

$$\begin{aligned} P-Q = {} & n^3(4\alpha^3 + 10\alpha^2 + 2\alpha + 4) + n^2(-5\alpha^4 - 30\alpha^3 - 55\alpha^2 - 34\alpha - 14) + \\ & + n(2\alpha^5+21\alpha^4+72\alpha^3+105\alpha^2+68\alpha+18) - 2\alpha^5 - 14\alpha^4 - 38\alpha^3 - 50\alpha^2 - 32\alpha - 8. \end{aligned}$$

Let E_n be $P-Q$. We will show that E_n it is a strictly increasing function.
We know that $\beta > \alpha \geq 2 \Longrightarrow n > 2\alpha+1$ $(n \geq 6)$

$$\begin{aligned} E''_n = {} & 6n(4\alpha^3 + 10\alpha^2 + 2\alpha + 4) - 10\alpha^4 - 60\alpha^3 - 110\alpha^2 - 68\alpha - 28 > \\ & > 6(2\alpha+1)(4\alpha^3 + 10\alpha^2 + 2\alpha + 4) - 10\alpha^4 - 60\alpha^3 - 110\alpha^2 - 68\alpha - 28 = \\ & = 38\alpha^4 + 84\alpha^3 - 26\alpha^2 - 8\alpha - 4 > 0. \end{aligned}$$

So, $\forall\, n \geq 6$, $E''_n > 0$, whence E'_n is a strictly increasing function. Since $n > 2\alpha+1$ we have $E'_n > E'_{2\alpha+1}$, therefore

$$\begin{aligned} E'_n > {} & 3(4\alpha^2 + 4\alpha + 1)(4\alpha^3 + 10\alpha^2 + 2\alpha + 4) - \\ & -2(2\alpha+1)(5\alpha^4 + 30\alpha^3 + 55\alpha^2 + 34\alpha + 14) + \\ & +2\alpha^5 + 21\alpha^4 + 72\alpha^3 + 105\alpha^2 + 68\alpha + 18 = \\ & = 30\alpha^5 + 59\alpha^4 - 52\alpha^3 - 39\alpha^2 - 2\alpha + 2 > 0. \end{aligned}$$

So E_n is a strictly increasing function. By consequence, $E_n > E_{2\alpha+1}$, that is

$$\begin{aligned} P - Q > &(8\alpha^3 + 12\alpha^2 + 6\alpha + 1)(4\alpha^3 + 10\alpha^2 + 2\alpha + 4)+ \\ &+(4\alpha^2 + 4\alpha + 1)(-5\alpha^4 - 30\alpha^3 - 55\alpha^2 - 34\alpha - 14)+ \\ &+(2\alpha + 1)(2\alpha^5 + 21\alpha^4 + 72\alpha^3 + 105\alpha^2 + 68\alpha + 18)- \\ &-2\alpha^5 - 14\alpha^4 - 38\alpha^3 - 50\alpha^2 - 32\alpha - 8 = \\ &= 16\alpha^6 + 30\alpha^5 - 34\alpha^4 - 22\alpha^3 + 14\alpha^2 + 8\alpha > 0. \end{aligned}$$

Then

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i)$$

Moreover we have

$$\begin{aligned} A(b_1) &= \frac{2(\beta-1)}{\beta} + \frac{2\alpha(\beta-1)}{n-1} + \frac{2(\beta-1)}{n} + 1 + \frac{2\alpha}{\alpha+1} + \frac{2}{\alpha+2} = \\ &= \frac{\begin{array}{c}(2n\alpha^2 + 6n\alpha + 4n - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8)\cdot \\ \cdot(2n^2 + n^2\alpha - \alpha^2 n - 2n\alpha - 3n + \alpha + 1)+ \\ +(3\alpha^2 + 9\alpha + 4)(n^3 - 2n^2 - n^2\alpha + n\alpha + n)\end{array}}{(\alpha+1)(\alpha+2)n(n-1)(n-\alpha-1)} \\ q(b_1) &= 2(\beta-1) + 2\alpha(\beta-1) + 2(\beta-1) + 1 + 2\alpha + 2 = \\ &= 2n\alpha + 4n - 2\alpha^2 - 6\alpha - 5. \end{aligned}$$

We have also, for $j > 1$,

$$A(b_j) = \beta - 1 + \frac{2(\beta-1)}{\beta} + \frac{2\alpha(\beta-1)}{n-1} + \frac{2(\beta-1)}{n} = (\beta-1)\left(\frac{2}{n} + \frac{2\alpha}{n-1} + \frac{2}{\beta} + 1\right)$$

$$q(b_j) = (\beta-1)^2 + 2(\beta-1) + 2\alpha(\beta-1) + 2(\beta-1) = (\beta-1)(n+\alpha+2),$$

so

$$\begin{aligned} \widetilde{\mu}_1(b_j) &= \frac{1}{n+\alpha+2}\left(\frac{2}{n} + \frac{2\alpha}{n-1} + \frac{2}{n-\alpha-1} + 1\right) = \\ &= \frac{n^3 + n^2\alpha + 2n^2 - 2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2}{n(n-1)(n-\alpha-1)(n+\alpha+2)}. \end{aligned}$$

Therefore $\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j)$ if and only if

$$\frac{\begin{array}{c}3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - 35n\alpha^2 - 42n\alpha \\ -14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(2n\alpha + 2n - \alpha^2 - 2\alpha - 2)} >$$

$$> \frac{n^3 + n^2\alpha + 2n^2 - 2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2}{n(n-1)(n-\alpha-1)(n+\alpha+2)}$$

if and only if

$$\begin{aligned} &P = (n^2 + n - \alpha^2 - 3\alpha - 2)(3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - \\ &-13n\alpha^3 - 35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8) > \\ &> Q = (2n\alpha^3 + 2n\alpha^2 - \alpha^4 - 2\alpha^3 - 2\alpha^2 + 6n\alpha^2 + 6n\alpha - 3\alpha^3 - 6\alpha^2 - 6\alpha + \\ &+4n\alpha + 4n - 2\alpha^2 - 4\alpha - 4)(n^3 + n^2\alpha + 2n^2 - 2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2). \end{aligned}$$

We obtain

$$P = n^4(3\alpha^3 + 15\alpha^2 + 22\alpha + 6) + n^3(-2\alpha^4 - 10\alpha^3 - 20\alpha^2 - 20\alpha - 8)+$$
$$+n^2(-3\alpha^5 - 26\alpha^4 - 84\alpha^3 - 127\alpha^2 - 88\alpha - 18) + n(2\alpha^6 + 19\alpha^5 + 79\alpha^4+$$
$$+175\alpha^3 + 220\alpha^2 + 142\alpha + 36) - 2\alpha^5 - 16\alpha^4 - 50\alpha^3 - 76\alpha^2 - 56\alpha - 16$$

and

$$Q = (2n\alpha^3 + 8n\alpha^2 + 10n\alpha + 4n - \alpha^4 - 5\alpha^3 - 10\alpha^2 - 10\alpha - 4)(n^3 + n^2\alpha + 2n^2-$$
$$-2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2) = n^4(2\alpha^3 + 8\alpha^2 + 10\alpha + 4) + n^3(\alpha^4 + 7\alpha^3 + 16\alpha^2+$$
$$+14\alpha+4)+n^2(-5\alpha^5-29\alpha^4-74\alpha^3-108\alpha^2-86\alpha-28)+n(2\alpha^6+13\alpha^5+44\alpha^4+$$
$$+95\alpha^3 + 124\alpha^2 + 90\alpha + 28) - 2\alpha^5 - 12\alpha^4 - 30\alpha^3 - 40\alpha^2 - 28\alpha - 8$$

whence,

$$P - Q = n^4(\alpha^3 + 7\alpha^2 + 12\alpha + 2) + n^3(-3\alpha^4 - 17\alpha^3 - 36\alpha^2 - 34\alpha - 12)+$$
$$+n^2(2\alpha^5 + 3\alpha^4 - 10\alpha^3 - 19\alpha^2 - 2\alpha + 10) + n(6\alpha^5 + 34\alpha^4 + 80\alpha^3 + 96\alpha^2+$$
$$+52\alpha + 18) - 4\alpha^4 - 20\alpha^3 - 36\alpha^2 - 28\alpha - 8.$$

Set $P - Q = E_n$. We have

$$E_n''' = 24n(\alpha^3 + 7\alpha^2 + 12\alpha + 2) - 18\alpha^4 - 102\alpha^3 - 216\alpha^2 - 204\alpha - 72 >$$
$$> (48\alpha+24)(\alpha^3+7\alpha^2+12\alpha+2)-18\alpha^4-102\alpha^3-216\alpha^2-204\alpha-72=$$
$$= 30\alpha^4 + 258\alpha^3 + 528\alpha^2 + 180\alpha - 24 > 0,$$

whence E_n'' is a strictly increasing function, so $\forall\, n > 2\alpha + 1$ we have $E_n'' > E_{2\alpha+1}''$. Since

$$E_n'' > 12(4\alpha^2 + 4\alpha + 1)(\alpha^3 + 7\alpha^2 + 12\alpha + 2)+$$
$$+ 6(2\alpha + 1)(-3\alpha^4 - 17\alpha^3 - 36\alpha^2 - 34\alpha - 12)+$$
$$+ 4\alpha^5 + 6\alpha^4 - 20\alpha^3 - 38\alpha^2 - 4\alpha + 20 =$$
$$= 16\alpha^5 + 168\alpha^4 + 370\alpha^3 + 94\alpha^2 - 112\alpha - 28 > 0,$$

it follows E_n' is a strictly increasing function, so $E_n' > E_{2\alpha+1}'$. We have

$$E_n' > 4(8\alpha^3 + 12\alpha^2 + 6\alpha + 1)(\alpha^3 + 7\alpha^2 + 12\alpha + 2)+$$
$$+ 3(4\alpha^2 + 4\alpha + 1)(-3\alpha^4 - 17\alpha^3 - 36\alpha^2 - 34\alpha - 12)+$$
$$+ (4\alpha + 2)(2\alpha^5 + 3\alpha^4 - 10\alpha^3 - 19\alpha^2 - 2\alpha + 10)+$$
$$+ 6\alpha^5 + 34\alpha^4 + 80\alpha^3 + 96\alpha^2 + 52\alpha + 8 =$$
$$= 4\alpha^6 + 54\alpha^5 + 99\alpha^4 - 95\alpha^3 - 198\alpha^2 - 62\alpha > 0,$$

therefore E_n is a strictly increasing, and thus $\forall\, n \geq 2\alpha + 2$ we have $E_n \geq E_{2\alpha+2}$.

Then

$$
\begin{aligned}
E_n \geq\ & (16\alpha^4 + 64\alpha^3 + 96\alpha^2 + 64\alpha + 16)(\alpha^3 + 7\alpha^2 + 12\alpha + 2)+\\
&+ (8\alpha^3 + 24\alpha^2 + 24\alpha + 8)(-3\alpha^4 - 17\alpha^3 - 36\alpha^2 - 34\alpha - 12)+\\
&+ (4\alpha^2 + 8\alpha + 4)(2\alpha^5 + 3\alpha^4 - 10\alpha^3 - 19\alpha^2 - 2\alpha + 10)+\\
&+ (2\alpha + 2)(6\alpha^5 + 34\alpha^4 + 80\alpha^3 + 96\alpha^2 + 52\alpha + 8)-\\
&- 4\alpha^4 - 20\alpha^3 - 36\alpha^2 - 28\alpha - 8 =\\
&= 8\alpha^6 + 40\alpha^5 + 48\alpha^4 - 36\alpha^3 - 112\alpha^2 - 76\alpha - 32 > 0.
\end{aligned}
$$

By consequence $\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j)$.

On the other side, $\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1)$ if and only if

$$
\frac{\begin{array}{c}3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - 35n\alpha^2 - 4n\alpha-\\ -14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(2n\alpha + 2n - \alpha^2 - 2\alpha - 2)} >
$$

$$
> \frac{\begin{array}{c}(2n\alpha^2 + 6n\alpha + 4n - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8)(2n^2 + n^2\alpha - n\alpha^2 - 2n\alpha-\\ -3n + \alpha + 1) + (3\alpha^2 + 9\alpha + 4)(n^3 - 2n^2 - n^2\alpha + n\alpha + n)\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(n-\alpha-1)(2n\alpha + 4n - 2\alpha^2 - 6\alpha - 5)}.
$$

Set $P = (2n^2\alpha + 4n^2 - 2n\alpha^2 - 6n\alpha - 5n - 2n\alpha^2 - 4n\alpha + 2\alpha^3 + 6\alpha^2 + 5\alpha - 2n\alpha - 4n + 2\alpha^2 + 6\alpha + 5)(3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - 35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8)$ and
$Q = (2n\alpha + 2n - \alpha^2 - 2\alpha - 2)(2n^3\alpha^3 + 13n^3\alpha^2 + 25n^3\alpha + 12n^3 - 4n^2\alpha^4 - 27n^2\alpha^3 - 73n^2\alpha^2 - 88n^2\alpha - 36n^2 + 2n\alpha^5 + 14n\alpha^4 + 47n\alpha^3 + 90n\alpha^2 + 87n\alpha + 32n - 2\alpha^4 - 12\alpha^3 - 26\alpha^2 - 24\alpha - 8)$.
We obtain

$$
\begin{aligned}
P =\ & n^4(6\alpha^4 + 42\alpha^3 + 104\alpha^2 + 100\alpha + 24) + n^3(-16\alpha^5 - 130\alpha^4 - 417\alpha^3-\\
&- 647\alpha^2 - 466\alpha - 110) + n^2(14\alpha^6 + 130\alpha^5 + 515\alpha^4 + 1101\alpha^3 + 1312\alpha^2+\\
&+ 802\alpha + 188) + n(-4\alpha^7 - 42\alpha^6 - 204\alpha^5 - 518\alpha^4 - 1016\alpha^3 - 1063\alpha^2-\\
&- 604\alpha - 142) + 4\alpha^6 + 36\alpha^5 + 134\alpha^4 + 264\alpha^3 + 290\alpha^2 + 168\alpha + 40
\end{aligned}
$$

and

$$
\begin{aligned}
Q =\ & n^4(4\alpha^4 + 30\alpha^3 + 76\alpha^2 + 74\alpha + 24)+\\
&+ n^3(-10\alpha^5 - 79\alpha^4 - 255\alpha^3 - 430\alpha^2 - 342\alpha - 96)+\\
&+ n^2(8\alpha^6 + 67\alpha^5 + 257\alpha^4 + 562\alpha^3 + 732\alpha^2 + 506\alpha + 136)+\\
&+ n(-2\alpha^7 - 18\alpha^6 - 83\alpha^5 - 240\alpha^4 - 437\alpha^3 - 486\alpha^2 - 302\alpha - 72)+\\
&+ 2\alpha^6 + 16\alpha^5 + 54\alpha^4 + 124\alpha^3 + 108\alpha^2 + 64\alpha + 16.
\end{aligned}
$$

So

$$\begin{aligned}P-Q=&n^4(2\alpha^4+12\alpha^3+28\alpha^2+26\alpha)+\\&+n^3(-6\alpha^5-51\alpha^4-162\alpha^3-217\alpha^2-124\alpha-14)+\\&+n^2(6\alpha^6+63\alpha^5+258\alpha^4+539\alpha^3+580\alpha^2+296\alpha+52)+\\&+n(-2\alpha^7-24\alpha^6-121\alpha^5-341\alpha^4-579\alpha^3-577\alpha^2-302\alpha-70)+\\&+2\alpha^6+20\alpha^5+80\alpha^4+140\alpha^3+182\alpha^2+104\alpha+34.\end{aligned}$$

For $n>2\alpha+1$ and $\alpha\geq 2$ we have

$$\begin{aligned}&P-Q>(16\alpha^4+32\alpha^3+24\alpha^2+8\alpha+1)(2\alpha^4+12\alpha^3+28\alpha^2+26\alpha)+\\&+(8\alpha^3+12\alpha^2+6\alpha+1)(-6\alpha^5-51\alpha^4-162\alpha^3-217\alpha^2-124\alpha-14)+\\&+(4\alpha^2+4\alpha+1)(6\alpha^6+63\alpha^5+258\alpha^4+539\alpha^3+580\alpha^2+296\alpha+52)+\\&+(2\alpha+1)(-2\alpha^7-24\alpha^6-121\alpha^5-341\alpha^4-579\alpha^3-577\alpha^2-302\alpha-70)+\\&+2\alpha^6+20\alpha^5+80\alpha^4+140\alpha^3+182\alpha^2+104\alpha+34.\end{aligned}$$

Therefore we find

$$P-Q>14\alpha^8+2\alpha^7-38\alpha^6+92\alpha^5+238\alpha^4+246\alpha^3+80\alpha^2-16\alpha+2>0.$$

It follows

$$\widetilde{\mu}_1(a_i)>\widetilde{\mu}_1(b_1)$$

Therefore we have obtained

$$\widetilde{\mu}_1(e)>\widetilde{\mu}_1(a_i)>\widetilde{\mu}_1(b_1),\quad \widetilde{\mu}_1(a_i)>\widetilde{\mu}_1(b_j)$$

For $j>1$ we have $\widetilde{\mu}_1(b_j)<\widetilde{\mu}_1(b_1)$ if and only if

$$\frac{n^3+n^2\alpha+2n^2-2n\alpha^2-3n\alpha-5n+2\alpha+2}{n(n-1)(n-\alpha-1)(n+\alpha+2)}<$$

$$<\frac{\begin{array}{c}(2n\alpha^2+6n\alpha+4n-2\alpha^3-10\alpha^2-16\alpha-8)\cdot\\ \cdot(2n^2+n^2\alpha-\alpha^2n-2n\alpha-3n+\alpha+1)+\\ +(3\alpha^2+9\alpha+4)(n^3-2n^2-n^2\alpha+n\alpha+n)\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(n-\alpha-1)(2n\alpha+4n-2\alpha^2-6\alpha-5)}$$

We set again

$$\begin{aligned}P=&(2n\alpha+4n-2\alpha^2-6\alpha-5)(n^3\alpha^2+3n^3\alpha+2n^3+n^2\alpha^3+5n^2\alpha^2+8n^2\alpha+\\&+4n^2-2n\alpha^4-9n\alpha^3-18n\alpha^2-21n\alpha-10n+2\alpha^3+8\alpha^2+10\alpha+4)\end{aligned}$$

whence $P=n^4(2\alpha^3+10\alpha^2+16\alpha+8)+n^3(2\alpha^3+9\alpha^2+13\alpha+6)+n^2(-6\alpha^5-42\alpha^4-123\alpha^3-195\alpha^2-168\alpha-60)+n(4\alpha^6+30\alpha^5+104\alpha^4+219\alpha^3+288\alpha^2+213\alpha+66)-4\alpha^5-28\alpha^4-78\alpha^3-108\alpha^2-74\alpha-20$ and

$$\begin{aligned}Q=&(n+\alpha+2)(2n^3\alpha^3+13n^3\alpha^2+25n^3\alpha+12n^3-4n^2\alpha^4-27n^2\alpha^3-73n^2\alpha^2-88n^2\alpha-36n^2+\\&+2n\alpha^5+14n\alpha^4+47n\alpha^3+90n\alpha^2+87n\alpha+32n-2\alpha^4-12\alpha^3-26\alpha^2-24\alpha-8),\end{aligned}$$

whence $Q = n^4(2\alpha^3 + 13\alpha^2 + 25\alpha + 12) + n^3(-2\alpha^4 - 10\alpha^3 - 22\alpha^2 - 26\alpha - 12) +$
$+ n^2(-2\alpha^5 - 21\alpha^4 - 80\alpha^3 - 144\alpha^2 - 125\alpha - 40) + n(2\alpha^6 + 18\alpha^5 + 73\alpha^4 +$
$+ 172\alpha^3 + 241\alpha^2 + 182\alpha + 56) - 2\alpha^5 - 16\alpha^4 - 50\alpha^3 - 76\alpha^2 - 56\alpha - 16.$
Hence

$$P - Q = n^4(-3\alpha^2 - 9\alpha - 4) + n^3(2\alpha^4 + 12\alpha^3 + 31\alpha^2 + 39\alpha + 18) +$$
$$+ n^2(-4\alpha^5 - 21\alpha^4 - 43\alpha^3 - 51\alpha^2 - 43\alpha - 20) + n(2\alpha^6 + 12\alpha^5 + 31\alpha^4 +$$
$$+ 47\alpha^3 + 47\alpha^2 + 31\alpha + 10) - 2\alpha^5 - 12\alpha^4 - 28\alpha^3 - 32\alpha^2 - 18\alpha - 4.$$

Set $\alpha = 3$. We have

$$\text{for } n = 8,\ P - Q > 0, \text{ whence } \widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(b_1)$$
$$\text{for } n \geq 9,\ P - Q < 0, \text{ whence } \widetilde{\mu}_1(b_j) < \widetilde{\mu}(b_1).$$

Set $\alpha = 4$. We have $P - Q < 0$ if and only if $n \geq 13$, so

$$\text{for } n < 13,\ \widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(b_1) \ \text{ and } \ \text{for } n \geq 13,\ \widetilde{\mu}(b_j) < \widetilde{\mu}_1(b_1).$$

Set $\alpha = 5$. We have $P - Q < 0$ if and only if $n \geq 25$. So

$$\text{for } n < 25,\ \widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(b_1) \ \text{ and } \ \text{for } n \geq 25,\ \widetilde{\mu}_1(b_j) < \widetilde{\mu}_1(b_1)$$

The following problem remains open:
To find a function $n(\alpha)$ such that $P - Q < 0$ if and only if $n \geq n(\alpha)$.

Let us consider now the case $\alpha = 3$, $\beta = 4$, whence $n = 8$.
Set $H = A \cup B \cup \{e\}$, where $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$ and let us suppose $\forall\, i, \forall\, j \in B' = B - \{b_1\}$,

$$\widetilde{\mu}(e) > \widetilde{\mu}(a_i) > \widetilde{\mu}(b_j) > \widetilde{\mu}(b_1)$$

so we have ${}^1H$ as follows, where $H' = H \setminus \{b_1\}$.

${}^1H$	b_1	b_2	b_3	b_4	a_1	a_2	a_3	e
b_1	b_1	B	B	B	A,B	A,B	A,B	H
b_2	B	B'	B'	B'	A,B'	A,B'	A,B'	H'
b_3	B	B'	B'	B'	A,B'	A,B'	A,B'	H'
b_4	B	B'	B'	B'	A,B'	A,B'	A,B'	H'
a_1	A,B	A,B'	A,B'	A,B'	A	A	A	A,e
a_2	A,B	A,B'	A,B'	A,B'	A	A	A	A,e
a_3	A,B	A,B'	A,B'	A,B'	A	A	A	A,e
e	H	H'	H'	H'	A,e	A,e	A,e	e

Therefore $A(b_1) = 1 + \dfrac{6}{4} + \dfrac{6}{7} + \dfrac{2}{8} = \dfrac{101}{28} \cdot q(b_1) = 15$, whence ${}^2\widetilde{\mu}(b_1) = 0,240$.
We have also $\forall\, j > 1$, $A(b_j) = \dfrac{6}{4} + \dfrac{6}{7} + \dfrac{2}{8} + \dfrac{9}{3} + \dfrac{18}{6} + \dfrac{6}{7} = \dfrac{265}{28}$, $q(b_j) = 47$

whence ${}^2\widetilde{\mu}(b_j) = 0,20136$. Moreover, $\forall\, i,\ A(a_i) = \frac{6}{7} + \frac{2}{8} + \frac{18}{6} + \frac{6}{7} + \frac{9}{3} + \frac{6}{4} = \frac{265}{28}$, $q(a_i) = 47$, whence ${}^2\widetilde{\mu}(a_i) = {}^2\widetilde{\mu}(b_j)$ and, finally,

$$A(e) = \frac{2}{8} + \frac{6}{7} + \frac{6}{4} + 1 = \frac{101}{28}, \quad q(e) = 15,$$

whence

$${}^2\widetilde{\mu}(e) = {}^2\widetilde{\mu}(b_1).$$

One obtains the second join space associated ${}^2H$, where $B' \cup A = C$.

${}^2H$	e	b_1	b_2	b_3	b_4	a_1	a_2	a_3
e	e, b_1	e, b_1	H	H	H	H	H	H
b_1	e, b_1	e, b_1	H	H	H	H	H	H
b_2	H	H	C	C	C	C	C	C
b_3	H	H	C	C	C	C	C	C
b_4	H	H	C	C	C	C	C	C
a_1	H	H	C	C	C	C	C	C
a_2	H	H	C	C	C	C	C	C
a_3	H	H	C	C	C	C	C	C

We have $A(e) = A(b_1) = \frac{4}{2} + \frac{24}{8} = 5$ and $q(e) = q(b_1) = 28$, whence ${}^3\widetilde{\mu}(b_1) = {}^3\widetilde{\mu}(e) = 0,178$. Moreover, $\forall\, j > 1,\ \forall\, i$,

$$A(b_j) = A(a_i) = \frac{24}{8} + \frac{36}{6} = 9, \quad q(b_j) = q(a_i) = 60,$$

whence ${}^3\widetilde{\mu}(b_j) = {}^3\widetilde{\mu}(a_i) = 0,15$. It follows ${}^3H = {}^2H$, hence

$$\forall\, r > 2,\ {}^rH = {}^2H \ \text{ and } \ s.f.g.(H) = 2.$$

3.3.6 Fuzzy grade of i.p.s. hypergroups

Corsini proved in [6] that any i.p.s. hypergroup with the cardinal smaller than 9 is strongly canonical.

For $n \leq 8$, have been determined by Corsini in [6], [7], [8] all the non-isomorphic i.p.s. hypergroups of cardinal n; their fuzzy grades have been studied by Corsini and Cristea in [18] and [19]. In the following we present the results obtained in this direction.

Theorem 3.3.26.

(i) *There is a unique i.p.s. hypergroup H of order 3 and $s.f.g.(H) = 1$.*

(ii) *There are three non isomorphic i.p.s. hypergroups H_1, H_2, H_3 of order 4 and $s.f.g.(H_1) = s.f.g.(H_2) = 1$, $s.f.g.(H_3) = 2$.*

(iii) *There are eight non isomorphic i.p.s. hypergroups of order* 5*: one of them has* $s.f.g.(H) = 2$ *and the other one have* $s.f.g.(H) = 1$.

(iv) *There are nineteen non isomorphic i.p.s. hypergroups of order* 6*: fourteen of them have* $s.f.g.(H) = 1$*, four of them are of* $s.f.g.(H) = 2$ *and one is with* $s.f.g.(H) = 3$.

(v) *There are thirty-six non isomorphic i.p.s. hypergroups of order* 7*: two of them have* $f.g.(H) = 1$*, twenty-five of them have* $s.f.g.(H) = 1$*, eight of them are with* $s.f.g.(H) = 2$ *and only for of them have* $s.f.g.(H) = 3$

The proof is detailed in the Appendix B.

3.3.7 The sequence of join spaces associated with a rough set

Rough sets theory, proposed by Zdzislaw Pawlak in 1982, has been attracting researchers and practitioners in various fields of science and technology. The interest in rough sets theory and applications is remarkable since the beginning and it is still growing. The ingenious concepts of rough set have been a base for original developments in both theoretical research, including logics, algebra and topology, and applied research, including knowledge discovery, data mining, decision theory, artificial intelligence. Latter have been found many applications of them in medicine, engineering, chemistry, bioinformatics, economy.

Let R be an equivalence relation on a universe H; for any element $x \in H$, we denote its equivalence class related to R by $R(x)$. For any non-empty subset A of H, *a rough set* of H is determined by the pair $(\underline{R}(A), \overline{R}(A))$, where

$$\underline{R}(A) = \bigcup_{R(x) \subset A} R(x) \quad \text{and} \quad \overline{R}(A) = \bigcup_{R(x) \cap A \neq \emptyset} R(x).$$

R.Biswas proved in 1999 (see [2]) that any rough set is a particular type of fuzzy set, but conversely not.

Later P.Corsini has associated with a universe H, endowed with a rough set, the following hyperoperation:

$$\forall (x, y) \in H^2, x \circ y = \overline{R}(\{x, y\}) \setminus \underline{R}(\{x, y\}).$$

Proposition 3.3.27. *The partial hyperoperation "$\circ$" is defined everywhere if and only if for any* $x \in H$, $\mid R(x) \mid \geq 3$.

Proof. Let us suppose there exists $x \in H$ such that $R(x) = \{x, x'\}$, with $x \neq x'$. Then

$$x \circ x' = \bigcup_{R(z) \cap \{x, x'\} \neq \emptyset} R(z) \setminus \bigcup_{R(z) \subset \{x, x'\}} R(z) = R(x) \setminus R(x) = \emptyset.$$

Let us suppose now there exists $x \in H$ such that $R(x) = \{x\}$. By the same way one finds $x \circ x = R(x) \setminus R(x) = \emptyset$. Therefore, for any $x \in H$, $|R(x)| \geq 3$.

We prove now the other implication. For any $x \in H$, set $|R(x)| \geq 3$. Then we have

$$\underline{R}(\{x,y\}) = \bigcup_{R(z)\subset\{x,y\}} R(z) = \emptyset,$$

whence $x \circ y = R(x) \cup R(y) \neq \emptyset$.

Proposition 3.3.28. *The hypergroupoid $\langle H, \circ \rangle$ is a join space if and only if, for any $x \in H$, $|R(x)| \geq 3$.*

Proof. By the previous proposition it is sufficient to prove that if $\langle H, \circ \rangle$ is a hypergroupoid, then it is a join space. We remark that the hypothesis $x \in H$, $|R(x)| \geq 3$ implies $x \circ y = R(x) \cup R(y)$. It follows that $\langle H, \circ \rangle$ is a commutative semihypergroup.

Moreover, since every x is an identity, it follows that the reproducibility law is satisfies. So $\langle H, \circ \rangle$ is a commutative hypergroup.

It remains to prove that the implication $a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset$ is satisfies.

Set $x \in a/b \cap c/d$, that is $a \in b \circ x$ and $c \in d \circ x$, whence

$$a \in R(b) \cup R(x), c \in R(d) \cup R(x).$$

We have $a \in a \circ d$, so, if $a \in R(b) \subset b \circ c$, it results $a \in a \circ d \cap b \circ c$.

By the same way $c \in R(d)$ implies $c \in a \circ d \cap b \circ c$.

We suppose now $a \notin R(b)$ and $c \notin R(d)$; it follows $a \in R(x)$, whence $x \in R(a) \subset a \circ d$ and $c \in R(x)$, whence $x \in R(c) \subset b \circ c$. Therefore $a \circ d \cap b \circ c \neq \emptyset$.

In the following we consider H a universe of order n endowed with a rough set.

We associate with this join space the fuzzy set $\widetilde{\mu}$ defined by the relations (ω) from the paragraph 3.3.1. We obtain, for any $x, y \in H$ and any $u \in x \circ y$,

$$\mu_{x,y}(u) = \begin{cases} \dfrac{1}{| R(x) |} & \text{, if } (x,y) \in R \\ \dfrac{1}{| R(x) | + | R(y) |} & \text{, if } (x,y) \notin R \end{cases}$$

We denote the factor set of H related to the relation R by

$$H/R = \{R(x_1), R(x_2), ..., R(x_n)\} \quad \text{with} \quad | R(x_i) | = \lambda_i.$$

For any $u \in H$, there exists $i \in \{1, 2, ..., n\}$ such that $u \in R(x_i)$ and then

$$\widetilde{\mu}(u) = \frac{1 + 2\left(\dfrac{\lambda_1}{\lambda_1+\lambda_i} + \dfrac{\lambda_2}{\lambda_2+\lambda_i} + ... + \dfrac{\lambda_{i-1}}{\lambda_{i-1}+\lambda_i} + \dfrac{\lambda_{i+1}}{\lambda_{i+1}+\lambda_i} + ... + \dfrac{\lambda_n}{\lambda_n+\lambda_i}\right)}{2n - \lambda_i}.$$

If $|R(u)| = |R(v)|$, then $\widetilde{\mu}(u) = \widetilde{\mu}(v)$.

Conversely, if $\widetilde{\mu}(u) = \widetilde{\mu}(v)$, then taking $u \in R(x_i)$ and $v \in R(x_j)$ with $i < j$ we obtain

$$\frac{1+2\left(\frac{\lambda_1}{\lambda_1+\lambda_i}+\frac{\lambda_2}{\lambda_2+\lambda_i}+...+\frac{\lambda_{i-1}}{\lambda_{i-1}+\lambda_i}+\frac{\lambda_{i+1}}{\lambda_{i+1}+\lambda_i}+...+\frac{\lambda_n}{\lambda_n+\lambda_i}\right)}{2n-\lambda_i} =$$

$$=\frac{1+2\left(\frac{\lambda_1}{\lambda_1+\lambda_j}+\frac{\lambda_2}{\lambda_2+\lambda_j}+...+\frac{\lambda_{j-1}}{\lambda_{j-1}+\lambda_j}+\frac{\lambda_{j+1}}{\lambda_{j+1}+\lambda_j}+...+\frac{\lambda_n}{\lambda_n+\lambda_j}\right)}{2n-\lambda_j} \Longleftrightarrow$$

$$2n-\lambda_j+2(2n-\lambda_j)\left(\frac{\lambda_1}{\lambda_1+\lambda_i}+...+\frac{\lambda_{i-1}}{\lambda_{i-1}+\lambda_i}+\frac{\lambda_{i+1}}{\lambda_{i+1}+\lambda_i}+...+\frac{\lambda_n}{\lambda_n+\lambda_i}\right) =$$

$$=2n-\lambda_i+2(2n-\lambda_i)\left(\frac{\lambda_1}{\lambda_1+\lambda_j}+...+\frac{\lambda_{j-1}}{\lambda_{j-1}+\lambda_j}+\frac{\lambda_{j+1}}{\lambda_{j+1}+\lambda_j}+...+\frac{\lambda_n}{\lambda_n+\lambda_j}\right).$$

Calculating, we obtain

$$(\lambda_i-\lambda_j)+4n(\lambda_j-\lambda_i)\left[\frac{\lambda_1}{(\lambda_1+\lambda_i)(\lambda_1+\lambda_j)}+...+\frac{\lambda_{i-1}}{(\lambda_{i-1}+\lambda_i)(\lambda_{i-1}+\lambda_j)}+\right.$$

$$+\frac{\lambda_{i+1}}{(\lambda_{i+1}+\lambda_i)(\lambda_{i+1}+\lambda_j)}+...+\frac{\lambda_{j-1}}{(\lambda_{j-1}+\lambda_i)(\lambda_{j-1}+\lambda_j)}+\frac{\lambda_{j+1}}{(\lambda_{j+1}+\lambda_i)(\lambda_{j+1}+\lambda_j)}+...+$$

$$\left.+\frac{\lambda_n}{(\lambda_n+\lambda_i)(\lambda_n+\lambda_j)}+\frac{1}{\lambda_i+\lambda_j}\right]+2(\lambda_i-\lambda_j)\left[\frac{\lambda_1(\lambda_1+\lambda_i+\lambda_j)}{(\lambda_1+\lambda_i)(\lambda_1+\lambda_j)}+...+\right.$$

$$+\frac{\lambda_{i-1}(\lambda_{i-1}+\lambda_i+\lambda_j)}{(\lambda_{i-1}+\lambda_i)(\lambda_{i-1}+\lambda_j)}+\frac{\lambda_{i+1}(\lambda_{i+1}+\lambda_i+\lambda_j)}{(\lambda_{i+1}+\lambda_i)(\lambda_{i+1}+\lambda_j)}+...+\frac{\lambda_{j-1}(\lambda_{j-1}+\lambda_i+\lambda_j)}{(\lambda_{j-1}+\lambda_i)(\lambda_{j-1}+\lambda_j)}+$$

$$\left.+\frac{\lambda_{j+1}(\lambda_{j+1}+\lambda_i+\lambda_j)}{(\lambda_{j+1}+\lambda_i)(\lambda_{j+1}+\lambda_j)}+...+\frac{\lambda_n(\lambda_n+\lambda_i+\lambda_j)}{(\lambda_n+\lambda_i)(\lambda_n+\lambda_j)}+1\right]=0.$$

It is equivalent with

$$(\lambda_i-\lambda_j)\left[1+\frac{2\lambda_1(\lambda_1+\lambda_i+\lambda_j-2n)}{(\lambda_1+\lambda_i)(\lambda_1+\lambda_j)}+...+\frac{2\lambda_{i-1}(\lambda_{i-1}+\lambda_i+\lambda_j-2n)}{(\lambda_{i-1}+\lambda_i)(\lambda_{i-1}+\lambda_j)}+\right.$$

$$+\frac{2\lambda_{i+1}(\lambda_{i+1}+\lambda_i+\lambda_j-2n)}{(\lambda_{i+1}+\lambda_i)(\lambda_{i+1}+\lambda_j)}+...+\frac{2\lambda_{j-1}(\lambda_{j-1}+\lambda_i+\lambda_j-2n)}{(\lambda_{j-1}+\lambda_i)(\lambda_{j-1}+\lambda_j)}+$$

$$\left.+\frac{2\lambda_{j+1}(\lambda_{j+1}+\lambda_i+\lambda_j-2n)}{(\lambda_{j+1}+\lambda_i)(\lambda_{j+1}+\lambda_j)}+...+\frac{2\lambda_n(\lambda_n+\lambda_i+\lambda_j-2n)}{(\lambda_n+\lambda_i)(\lambda_n+\lambda_j)}+\frac{2\lambda_i+2\lambda_j-4n}{\lambda_i+\lambda_j}\right]=0$$

Since all the term of the form $\dfrac{2\lambda_1(\lambda_1+\lambda_i+\lambda_j-2n)}{(\lambda_1+\lambda_i)(\lambda_1+\lambda_j)}$ is negative and also $1+\dfrac{2\lambda_i+2\lambda_j-4n}{\lambda_i+\lambda_j}<0$ we obtain that the previous relation is 0 if and only if $\lambda_i=\lambda_j$.

We conclude that $\widetilde{\mu}(u)=\widetilde{\mu}(v) \Longleftrightarrow |R(u)|=|R(v)|$.

Moreover, we have $\widetilde{\mu}(u)<\widetilde{\mu}(v) \Longleftrightarrow |R(u)|>|R(v)|$.

We observe that if R is an equivalence relation on H such that, for any $i\in\{1,2,...,n\}$, we have $\mid R(x_i)\mid = k = const$, then $\widetilde{\mu}(x)=\widetilde{\mu}(y)$, for any $x,y\in H$ and therefore the join space ${}^1H$ associated is a total hypergroup and $s.f.g.(H)=1$.

3.3.8 Properties of the join spaces ${}^iH$ associated with the hypergroupoid H

Besides the fundamental relation β defined on a hypergroup, can be defined other three relations, named also fundamental and introduced by Jantosciak in [29]. With this relations he introduced the notion of reduced hypergroup. In the following we find a necessary and sufficient condition for the join spaces ${}^iH$ in order to be reduced hypergroups.

First, we remind some definitions and results from the paper of Jantosciak.

Definition 3.3.29. (see [29]) Let x,y be arbitrary elements of H.

(i) x and y are called *operationally equivalent* if $x\circ a=y\circ a$ and $a\circ x=a\circ y$, for any $a\in H$;

(ii) x and y are called *inseparable* if for $a,b\in H$, $x\in a\circ b$ if and only if $y\in a\circ b$;

(iii) x and y are called *essentially indistinguishable* if x and y are operationally equivalent and inseparable.

Remark. (see [29]) The three relations, operational equivalence, inseparability and essential indistinguishability, denoted by $\sim_o$, $\sim_i$ and $\sim_e$ respectively, are equivalence relations on H.

For any $x\in H$, let $\widehat{x}_o$, $\widehat{x}_i$ and $\widehat{x}_e$, respectively denote the equivalence class of x respect to the relations $\sim_o,\sim_i$ and $\sim_e$.

Proposition 3.3.30. (see [29]) *Let be $x,y\in H$. Then*

(i) $\widehat{x}_o\ \widehat{y}_o=x\circ y$;

(ii) $\widehat{(x\circ y)}_i=x\circ y$,

(iii) $\widehat{x}_e=\widehat{x}_o\cap\widehat{x}_i$ *and* $\widehat{x}_e\ \widehat{y}_e=\widehat{(x\circ y)}_e=x\circ y$.

Definition 3.3.31. (see [29]) A hypergroup H is said to be *reduced* if its e-classes are singletons, that is, for any $x \in H$, $\widehat{x}_e = \{x\}$.

Remark. Any group is a reduced hypergroup.

Remark. A hypergroup H is reduced if, for any $x \in H, \widehat{x}_o = \{x\}$ or $\widehat{x}_i = \{x\}$.

Proposition 3.3.32. (see [30]) *For any hypergroup $(H, \circ)$, the quotient hypergroupoid $(H/\sim_e, \star)$ is a reduced hypergroup, where the hyperoperation $\star$ on $H/\sim_e$ is defined by*

$$\widehat{x}_e \star \widehat{y}_e = \{\widehat{z}_e \mid z \in x \circ y\}.$$

Proof. First note that, for all $a, b \in H$, it takes place the relation

$$a \circ b = \widehat{(a \circ b)}_e = \bigcup \{\widehat{z}_e \mid z \in a \circ b\} = \bigcup \{\widehat{z}_e \mid z \in \widehat{a}_e \circ \widehat{b}_e\} = \bigcup \widehat{a}_e \star \widehat{b}_e. \quad (1)$$

Suppose $\widehat{x}_e$ and $\widehat{y}_e$ are essential indistinguishable in $H/\sim_e$. Then $\widehat{x}_e$ and $\widehat{y}_e$ are operationally equivalent, thus $\widehat{x}_e \star \widehat{a}_e = \widehat{y}_e \star \widehat{a}_e$ for all $\widehat{a}_e \in H/\sim_e$. By (1), for every $a \in H$ we obtain

$$x \circ a = \bigcup \widehat{x}_e \star \widehat{a}_e = \bigcup \widehat{y}_e \star \widehat{a}_e = y \circ a.$$

Similarly, $a \circ x = a \circ y$. Hence x and y are operationally equivalent in H.

But $\widehat{x}_e$ and $\widehat{y}_e$ are also inseparable in $H/\sim_e$, therefore, for all $\widehat{a}_e, \widehat{b}_e \in H/\sim_e$, one has $\widehat{x}_e \in \widehat{a}_e \star \widehat{b}_e$ if and only if $\widehat{y}_e \in \widehat{a}_e \star \widehat{b}_e$. By (1), for every $a, b \in H$, $z \in a \circ b$ if and only if $\widehat{z}_e \in \widehat{a}_e \star \widehat{b}_e$. For $a, b \in H$, we have $x \in a \circ b$ if and only if $y \in a \circ b$. Hence x and y are inseparable in H.

Therefore, x and y are essential indistinguishable in H, that is $\widehat{x}_e = \widehat{y}_e$.

Theorem 3.3.33. (see [54]) *For all i, $i \geq 1$, the quotient hypergroup ${}^iH/R_{\widetilde{\mu}_i}$ is a reduced hypergroup.*

Proof. First, for any $x \in H$, we determine the equivalence class of x in the ${}^iH$ respect to the equivalence relation $\sim_e$ on ${}^iH$.

By the Definition 3.3.29 we have

$$\begin{aligned}
\widehat{x}_i &= \{y \in H \mid y \in a \circ_i b \Leftrightarrow x \in a \circ_i b, \quad \text{for} \quad a, b \in H\} = \\
&= \{y \in H \mid \widetilde{\mu}_{i-1}(y) \in [\widetilde{\mu}_{i-1}(a), \widetilde{\mu}_{i-1}(b)] \Leftrightarrow \widetilde{\mu}_{i-1}(x) \in [\widetilde{\mu}_{i-1}(a), \widetilde{\mu}_{i-1}(b)], a, b \in H\} = \\
&= \{y \in H \mid \widetilde{\mu}_{i-1}(y) = \widetilde{\mu}_{i-1}(x)\}
\end{aligned}$$

where

$$[\widetilde{\mu}_{i-1}(a), \widetilde{\mu}_{i-1}(b)] = \{c \in [0,1] \mid \widetilde{\mu}_{i-1}(a) \wedge \widetilde{\mu}_{i-1}(b) \leq c \leq \widetilde{\mu}_{i-1}(a) \vee \widetilde{\mu}_{i-1}(b)\}.$$

Then

$$\begin{aligned}
\widehat{x}_o &= \{y \in H \mid x \circ_i a = y \circ_i a, \quad \text{for any} \quad a \in H\} = \\
&= \{y \in H \mid [\widetilde{\mu}_{i-1}(x), \widetilde{\mu}_{i-1}(a)] = [\widetilde{\mu}_{i-1}(y), \widetilde{\mu}_{i-1}(a)], \quad \text{for any} \quad a \in H\} = \\
&= \{y \in H \mid \widetilde{\mu}_{i-1}(y) = \widetilde{\mu}_{i-1}(x)\}.
\end{aligned}$$

By consequence, for any $x \in H$, $\widehat{x}_e = \widehat{x}_i \cap \widehat{x}_o = \{y \in H \mid \widetilde{\mu}_{i-1}(y) = \widetilde{\mu}_{i-1}(x)\}$ in ${}^iH$ and therefore ${}^iH/\sim_e = {}^{i-1}H/R_{\widetilde{\mu}_{i-1}}$. By the Proposition 3.3.32 it follows that the quotient hypergroup ${}^{i-1}H/R_{\widetilde{\mu}_{i-1}}$ is a reduced hypergroup.

We can not say the same thing about the join spaces ${}^iH$ from the sequence associated with H, but we obtain the following results.

Theorem 3.3.34. (see [54]) *The join space ${}^1H$ is a reduced hypergroup if and only if the n-tuple associated with H is $(1, 1, ..., 1)$. Moreover, the join space ${}^1H$ is the unique join space of the sequence $(\langle {}^rH, \circ_r\rangle, \widetilde{\mu}_r)_{r\geq 1}$ corresponding to H which can be a reduced hypergroup.*

Proof. Assume that $({}^iH, \circ_i)$, $i > 1$, is a reduced hypergroup, that is, for any $x \in H$, $\widehat{x}_e = \{x\}$ and as we have already seen, $\widehat{x}_e = \{y \in H \mid \widetilde{\mu}_{i-1}(y) = \widetilde{\mu}_{i-1}(x)\}$. It follows that, for any $y \in H, y \neq x$, $\widetilde{\mu}_{i-1}(y) \neq \widetilde{\mu}_{i-1}(x)\}$ and therefore the n-tuple associated with the join space $({}^{i-1}H, \circ_{i-1})$ is $(1, 1, ..., 1)$.
Conversely the proof is clear.

If ${}^iH$ is a reduced hypergroup then for all $j < i - 1$ the n-tuple associated with the join space ${}^jH$ is $(1, 1, ..., 1)$ and by consequence all the join spaces ${}^jH$, $j < i - 1$, are isomorphic with H. Therefore, only the join space ${}^1H$ can be a reduced hypergroup.

3.4 Other examples of reduced hypergroups

In this section we present some properties of the three fundamental relations defined by Jantosciak in [29] and we give some examples of the reduced hypergroups.

Proposition 3.4.1.

(i) *The operational equivalence is regular, but in general not strongly regular.*

(ii) *The inseparability is not a regular relation.*

(iii) *The essential indistinguishability is regular, but in general not strongly regular.*

Proof. (i) We consider $x, y \in H$ such that $x \sim_o y$ and u arbitrary in H. Then $x \circ u = y \circ u$ and there exists $b = a \in y \circ u$ such that $a \sim_o b$ (since $\sim_o$ is reflexive). Therefore $\sim_o$ is regular.

Now we give an example of a hypergroup H on which the relation $\sim_o$ is not strongly regular.
Let $H = \{a, b, c\}$ be the following hypergroup

H	a	b	c
a	a	a	a, b, c
b	a	a	a, b, c
c	a, b, c	a, b, c	c

We observe that $a \sim_o b$ and we don't have $a \sim_o c$.
We suppose that $\sim_o$ is strongly regular, that is:

$$a \sim_o b \Longrightarrow \forall u \in H, \forall x \in a \circ u, \forall y \in b \circ u : x \sim_o y.$$

For $u = c, x = a$ and $y = c$ it would follow $a \sim_o c$, which is a contradiction.

(ii) Let $H = \{a, b, c\}$ be the following hypergroupoid

H	a	b	c
a	a	a	a, b, c
b	a	a	b, c
c	a, b, c	b, c	b, c

We suppose that the relation $\sim_i$ is regular, so from $b \sim_i c$ it results $\forall u \in H, \forall x \in b \circ u, \exists y \in c \circ u$ such that $x \sim_i y$. For $u = b$ and $x = a$ we obtain that there exists $y \in \{b, c\}$ such that $a \sim_i b$ or $a \sim_i c$, which is a contradiction.

(iii) We prove that the relation $\sim_e$ is regular, that is,
$a \sim_e b \Longrightarrow \forall u \in H, \forall x \in a \circ u, \exists y \in b \circ u : x \sim_e y$.
From $a \sim_e b$ we have $a \sim_o b$, so $\forall u \in H, a \circ u = b \circ u$. Then, there exists $y = x \in b \circ u$ such that $x \sim_e y$ (since the relation $\sim_e$ is reflexive).

Now we give an example of a hypergroup H on which the relation $\sim_e$ is not strongly regular.
Let $H = \{a, b, c\}$ be the following hypergroup

H	a	b	c
a	a, b	a, b	a, b, c
b	a, b	a, b	a, b, c
c	a, b, c	a, b, c	a, b, c

We easily observe that $a \sim_e b$ and we suppose $\sim_e$ is strongly regular, then $\forall u \in H, \forall x \in a \circ u$, $\forall y \in b \circ u$ it results $x \sim_e y$. For $u = c, x = a, y = c$ it follows $a \sim_e c$, which is a contradiction.

Proposition 3.4.2. *If H is a hypergroup with a scalar identity, then, for $a, b \in H$, we have:*

$$x \sim_o y \Longleftrightarrow x \sim_i y \Longleftrightarrow x \sim_e y \Longleftrightarrow x = y.$$

(In this case all the three relations are regular).

Proof. Let e be a scalar identity in H.

Let us consider $a, b \in H$ such that $a \sim_o b$, that is $\forall x \in H, a \circ x = b \circ x$ and $x \circ a = x \circ b$. For $x = e$ we have $a = a \circ e = b \circ e = b$, so $a = b$.

Let us consider now $a, b \in H$ such that $a \sim_i b$, that is $a \in x \circ y \Longleftrightarrow b \in x \circ y$, for $x, y \in H$. But $a = a \circ e$, so $a \in a \circ e$ and thus $b \in a \circ e = a$. We obtain $a = b$.

Proposition 3.4.3. *Let $(H, \circ)$ be a hypergroupoid. The relation $\sim_i$ is regular on H if and only if $a \sim_i b \Longrightarrow a \sim_o b$, for $a, b \in H$.*

Proof. Let us suppose that the relation $\sim_i$ is regular on H.
We consider $a, b \in H$ such that $a \sim_i b$. We have to prove that $\forall u \in H, a \circ u = b \circ u$ and $u \circ a = u \circ b$.
For $x \in a \circ u$, by the regularity of $\sim_i$, there exists $y \in b \circ u$ such that $x \sim_i y$; since $x \sim_i y$ it follows $x \in b \circ u$, so $a \circ u \subset b \circ u$.
Similarly, $b \circ u \subset a \circ u$, so $a \circ u = b \circ u$, for any $u \in H$.
In the same way we prove $u \circ a = u \circ b$, for any $u \in H$.

Let's suppose now $a, b \in H$ with $a \sim_i b$, so $a \sim_o b$, by hypothesis. We show that $\forall u \in H, \forall x \in a \circ u, \exists y \in b \circ u$ such that $x \sim_i y$.
We take $u \in H$ and $x \in a \circ u$; it results that there exists $y = x \in b \circ u = a \circ u$ such that $x \sim_i y$. So, $\sim_i$ is regular on H.

Proposition 3.4.4. *Let $(H, \circ)$ be a hypergroup. If, for any $x \in H$, there exist $(a, b) \in H^2$ such that $a \circ b = x$, then H is a reduced hypergroup.*

Proof. We prove that for any $x \in H, \widehat{x_i} = \{x\}$ and therefore $\widehat{x_e} = \{x\}$.
Let x be an arbitrary element of H. We have

$$\widehat{x_i} = \{y \in H \mid x \in a \circ b \Longleftrightarrow y \in a \circ b, \quad \text{for} \quad a, b \in H\}.$$

Let's suppose that there exists $x_0 \in H$ such that $\mid \widehat{(x_0)_i} \mid \geq 2$, that is, there exists $y \in H \setminus \{x_0\}$ with $y \in \widehat{(x_0)_i}$; it follows that $y \in a \circ b = x_0$, contradiction.
Therefore, H is a reduced hypergroup.

Corollary 3.4.5. *If H is a hypergroup with scalar identities, then H is a reduced hypergroup.*

Example. Any i.p.s. hypergroup is a reduced hypergroup.

Proposition 3.4.6. *Let $(G, \cdot)$ be a commutative group and R an equivalence relation on G such that, for any $x \in G$, $\hat{x} = \{x, x^{-1}\}$. We define on G/R the hyperoperation*

$$\hat{x} \otimes \hat{y} = \{\widehat{xy}, \widehat{xy^{-1}}\}.$$

Then G/R is a reduced hypergroup.

Proof. Indeed, for any $x, y \in G$ we have $\widehat{xy^{-1}} = \{xy^{-1}, yx^{-1}\} = \widehat{yx^{-1}}$, thus $\hat{x} \otimes \hat{y} = \hat{y} \otimes \hat{x}$.

We verify the reproducibility law: for any $\hat{x} \in G/R, \hat{x} \otimes G/R = G/R$, that is, for any $\hat{y} \in G/R$, there exists $\hat{z} \in G/R$ such that $\hat{y} \in \hat{x} \otimes \hat{z}$. For $z = x^{-1}y$, we have $\hat{x} \otimes \hat{z} = \{\hat{y}, \widehat{xy^{-1}x}\} \ni \hat{y}$.

Now, we verify the associativity law.
For any $\hat{x}, \hat{y}, \hat{z} \in G/R$ we have:

$$\begin{array}{ll} (\hat{x} \otimes \hat{y}) \otimes \hat{z} & = (\widehat{xy} \otimes \hat{z}) \cup (\widehat{xy^{-1}} \otimes \hat{z}) = \\ & = \{\widehat{(xy)z}, \widehat{(xy)z^{-1}}, \widehat{(xy^{-1})z}, \widehat{(xy^{-1})z^{-1}}\} \\ \hat{x} \otimes (\hat{y} \otimes \hat{z}) & = (\hat{x} \otimes \widehat{yz}) \cup (\hat{x} \otimes \widehat{yz^{-1}}) = \\ & = \{\widehat{x(yz)}, \widehat{x(z^{-1}y^{-1})}, \widehat{x(yz^{-1})}, \widehat{x(zy^{-1})}\} \end{array}.$$

We obtain that $(\hat{x} \otimes \hat{y}) \otimes \hat{z} = \hat{x} \otimes (\hat{y} \otimes \hat{z})$, since G is a commutative group.

So, G/R is a commutative hypergroup.

For any $\hat{x} \in G/R$ we have $\hat{x} = \{x, x^{-1}\} = \widehat{x^{-1}}$ and $\hat{e} \otimes \hat{x} = \hat{x}$, where e is the identity of the group G. By the Proposition 3.4.4 it results that G/R is a reduced hypergroup.

Example 3.4.7. Let us consider $C(n) = \{e_0, e_1, ..., e_{k(n)}\}$, where $k(n) = \frac{n}{2}$ if $n \in 2\mathbb{N}$ and $k(n) = \frac{n-1}{2}$ if $n \in 2\mathbb{N} + 1$. For any $s, t \leq k(n)$ we define

$$e_s \circ e_t = \{e_p, e_r\}, \quad \text{where} \quad p = min\{s+t, n-(s+t)\}, r = \mid s - t \mid .$$

It is well known that $\langle C(n), \circ \rangle$ is a canonical hypergroup.
Moreover, for any $t \leq k(n)$, we have $e_0 \circ e_t = e_t$, because $r = \mid 0 - t \mid = t$ and $p = min\{t, n-t\} = t$. Therefore, by the Proposition 3.4.4, H is a reduced hypergroup.

Remark. The condition expressed in the Proposition 3.4.4 is a sufficient condition, but not a necessary one as we can see in the following example.

Example. Let $(H, \circ)$ be the following hypergroup

H	x	y	z
x	z	x, y	H
y	x, y	z	H
z	H	H	H

which is a commutative reduced hypergroup and doesn't verify the condition from the Proposition 3.4.4, because there exists $x \in H$ such that, for any $a, b \in H$ with $a \circ b \ni x$, we have $\mid a \circ b \mid \geq 2$.

Theorem 3.4.8. *Any join space $(H, \circ)$ obtained associating to a universe H endowed with a rough set $(\underline{R}(A), \overline{R}(A))$, the hyperoperation*

$$\forall (x, y) \in H^2, x \circ y = \overline{R}(\{x, y\}) \setminus \underline{R}(\{x, y\})$$

is not a reduced hypergroup.

Proof. It is known that the above hyperoperation is well defined if and only if for any $x \in H, \mid R(x) \mid \geq 3$, which is equivalent with the fact that $(H, \circ)$ is a join space.

To prove that H is not a reduced hypergroup we'll show that

$$xRy \Longleftrightarrow x \sim_e y, \quad \text{for} \quad x, y \in H.$$

Then $R(x) = \widehat{x_e}$ and thus $\forall x \in H, \widehat{x_e} \neq \{x\}$, so H is not reduced.

First we prove that $xRy \Longleftrightarrow x \sim_o y$, for $x, y \in H$.
Let x, y be arbitrary elements of H such that xRy. For any $a \in H$ we have $x \circ a = R(x) \cup R(a) = R(y) \cup R(a) = y \circ a$, so $x \sim_o y$.
We consider now $x, y \in H$ with $x \sim_o y$, that is, for any $a \in H$, $x \circ a = y \circ a$. We have $x \circ x = y \circ x$ thus $R(x) = R(x) \cup R(y)$, so $R(y) \subset R(x)$. From $x \circ y = y \circ y$ it results $R(x) \subset R(y)$. It follows xRy.

It remains to prove that $xRy \Longleftrightarrow x \sim_i y$, for $x, y \in H$.
We suppose that xRy and let us consider $u, v \in H$ such that $x \in u \circ v = R(u) \cup R(v)$; it implies $x \in R(u)$ or $x \in R(v)$, equivalent with xRu or xRv, thus yRu or yRv, so $y \in R(u) \cup R(v) = u \circ v$.
Conversely, we have $x \in x \circ x = R(x)$ and $x \sim_i y$, thus $y \in x \circ x = R(x)$, that is, xRy.

Chapter 4

Concluding remarks and future work

Along the previous contributions in the theory of hypergroups and especially to establish new various connections between hypergroups and fuzzy sets, this thesis confirms, through the original results obtained in the third chapter, the fact that these associations which lead to new algebraic structures open a gate towards the large universe of the hyperstructures. Many problems from this domain are still opened; we present below a few of them which may be a continuation of this research.

The central part of this thesis is formed by the Chapter 3, where we study the sequence of join-spaces and fuzzy sets associated with a hypergroupoid H.

We remember that, with any hypergroupoid H endowed with a fuzzy set, we can an ordered r-tuple $(k_1, k_2, ..., k_r)$, where $\sum_{i=1}^{r} k_i = n = \mid H \mid$, as in the subsection § 3.3.1.

The situation when the r-tuple has the form $(k, k, ..., k)$, $k \geq 1$, is discussed and solved by the Theorem 3.3.10; if the r-tuple is equal with its opposite one, that is $(k_1, k_2, ..., k_r) = (k_r, k_{r-1}, ..., k_1)$, then we can use the Theorem 3.3.15 to determine the length of the sequence of join spaces associated with H.

Problem 1. To study the join space ${}^1H$ associated with the hypergroupoid H, for the r-tuple $(1, 2, ..., r)$.

Problem 2. To verify if, for any $k \geq 1$, there exist a hypergroup H such that the fuzzy set $\widetilde{\mu}$ defined by the relations (ω) from the subsection § 3.3.1 determines exactly the r-tuple $(k, k, ..., k)$.

Till now, it is proved the case $k = 2$, considering on the universe H the hyperproduct:

$$a_i \circ a_i = a_i, a_i \circ a_j = \{a_i, a_{i+1}, ..., a_j\}, \quad \text{for} \quad i < j$$

(see the Corollary 3.3.14).

One of the fundamental results obtained in this paper is the Theorem 3.3.15, where we give a sufficient condition, but not a necessary one, such that two consecutive join spaces from the sequence associated with a hypergroupoid H are not isomorphic.

Problem 3. To find a necessary condition such that this join spaces are not isomorphic, that is the sequence associated with H does not end.

To any hypergroupoid corresponds a fuzzy grade, that is the length of the sequence of its associated join spaces. Given a natural number $n \in \mathbb{N}^* \setminus \{1\}$, we can construct a hypergroup H (which is a join space) such that $s.f.g.(H) = n$(see the Corollary 3.3.14).

Problem 4. To find a method which constructs the hypergroup H with the smaller cardinality such that $s.f.g.(H) = n$.

Associating to an universe H a rough set $(\underline{R}(A), \overline{R}(A))$, we can construct a join space $\langle H, \circ \rangle$, where, for any $x, y \in H$, $x \circ y = \overline{R}(\{x, y\}) \setminus \underline{R}(\{x, y\})$, if and only if for any $x \in H$, $\mid R(x) \mid \geq 3$. Now associating the fuzzy set $\widetilde{\mu}$ defined by the relations (ω) of the paragraph 3.3.1, we demonstrated if R is an equivalence relation such that $\mid R(x) \mid = k = const$, for any $x \in H$, then $s.f.g.(H) = 1$.

Problem 5. To study the fuzzy grade of the join space $\langle H, \circ \rangle$ associated with a rough set $(\underline{R}(A), \overline{R}(A))$, where R is an equivalence relation different by the one considered in the paragraph §3.3.6.

Till now, we have considered only some remarkable classes of hypergroups: the i.p.s. hypergroups, the complete hypergroups or the hypergroups endowed with a rough set.

Problem 6. To study the sequence associated with other important types of hypergroups, for example: the cyclic hypergroups, the hypergroups obtained from a binary relation, or from a hypergraph.

Appendix A

The sequence of fuzzy sets and of join spaces determined by all the complete hypergroups of order less than or equal to 6

Theorem 3.3.20. (see [17]) *Let H be a complete hypergroup of order $n \leq 6$.*

(i) *For $n = 3$, $s.f.g.(H) = 1$.*

(ii) *For $n = 4$, there are 3 hypergroups with $s.f.g.(H) = 1$ and two hypergroups with $s.f.g.(H) = 2$.*

(iii) *For $n = 5$, $s.f.g.(H) = 1$.*

(iv) *For $n = 6$, there are 17 hypergroups of $s.f.g.(H) = 1$ and 4 hypergroups of $s.f.g.(H) = 2$.*

Proof. (i) Let consider $H = \{e, a_1, a_2\}$ a complete hypergroup. In this case $G = (\mathbb{Z}_2, +)$, so we have two structures:

a)

H	e	a_1	a_2
e	e	A_1	A_1
a_1		e	e
a_2			e

where $A_0 = \{e\}$, $A_1 = \{a_1, a_2\}$. We note that H is an 1-hypergroup. By Theorem 3.3.17, $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_1) = \widetilde{\mu}(a_2) = 0,5$.

The join space ${}^1H$ associated is

${}^1H$	e	a_1	a_2
e	e	H	H
a_1		A_1	A_1
a_2			A_1

hence $\widetilde{\mu}_1(e) = 0,47$, $\widetilde{\mu}_1(a_1) = \widetilde{\mu}_1(a_2) = 0,42$. Then ${}^rH = {}^1H$, $\forall r \geq 2$.

b)

H	e	a_1	a_2
e	A_0	A_0	A_1
a_1		A_0	A_1
a_2			A_0

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$. We have $\widetilde{\mu}(e) = \widetilde{\mu}(a_1) = 0,5$ and $\widetilde{\mu}(a_2) = 1$.

It follows that the join space ${}^1H$ associated is

H	e	a_1	a_2
e	A_0	A_0	H
a_1		A_0	H
a_2			a_2

with $\widetilde{\mu}_1(e) = \widetilde{\mu}_1(a_1) = 0,417$, $\widetilde{\mu}_1(a_2) = 0,467$, so ${}^rH = {}^1H$, $\forall r \geq 2$.

(ii) Let $H = \{e, a_1, a_2, a_3\}$ be a complete hypergroup.

a) Setting $G = (\mathbb{Z}_2, +)$ we obtain:

a$_1$)

H	e	a_1	a_2	a_3
e	e	A_1	A_1	A_1
a_1		e	e	e
a_2			e	e
a_3				e

where $A_0 = \{e\}$, $A_1 = \{a_1, a_2, a_3\}$. H is an 1-hypergroup. The membership function is $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_i) = 0,(3)$, $1 \leq i \leq 3$.

The join space associated is:

${}^1H$	e	a_1	a_2	a_3
e	e	H	H	H
a_1		A_1	A_1	A_1
a_2			A_1	A_1
a_3				A_1

for which $\widetilde{\mu}_1(e) = 0,36$, $\widetilde{\mu}_1(a_i) = 0,3$, $1 \leq i \leq 3$, hence ${}^rH = {}^1H$, $\forall r \geq 2$.

a$_2$)

H	e	a_1	a_2	a_3
e	A_0	A_0	A_1	A_1
a_1		A_0	A_1	A_1
a_2			A_0	A_0
a_3				A_0

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2, a_3\}$. We obtain $\widetilde{\mu}(e) = \widetilde{\mu}(a_i)$, $1 \leq i \leq 3$ and it follows that ${}^1H$ is a total hypergroup and $s.f.g.(H) = 1$.

a$_3$)

H	e	a_1	a_2	a_3
e	A_0	A_0	A_0	A_1
a_1		A_0	A_0	A_1
a_2			A_0	A_1
a_3				A_0

with $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3\}$. Then $\widetilde{\mu}(e) = \widetilde{\mu}(a_1) = \widetilde{\mu}(a_2) = 0,(3)$ and $\widetilde{\mu}(a_3) = 1$. The join space associated is isomorphic with the one of a$_1$), hence ${}^rH = {}^1H$, $\forall r \geq 2$.

b) For $G = (\mathbb{Z}_3, +)$ we distinguish two hypergroups:

b$_1$)

H	e	a_1	a_2	a_3
e	e	a_1	A_2	A_2
a_1		A_2	e	e
a_2			a_1	a_1
a_3				a_1

where $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2, a_3\}$. H is an 1-hypergroup.

b$_2$)

H	e	a_1	a_2	a_3
e	A_0	A_0	a_2	a_3
a_1		A_0	a_2	a_3
a_2			a_3	A_0
a_3				a_2

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3\}$.

In the last two cases, we obtain that $\mid H/R_{\widetilde{\mu}} \mid = 2$ and every class of equivalence has the same cardinal number, therefore the $s.f.g.(H) = 2$ and ${}^2H$ is a total hypergroup (since the Case **I** which follows the Definition 3.3.9).

iii) Set $H = \{e, a_1, a_2, a_3, a_4\}$ a complete hypergroup.

a) There are five complete 1-hypergroups of order 5.

a$_1$)

H	e	a_1	a_2	a_3	a_4
e	e	A_1	A_1	A_1	A_1
a_1		e	e	e	e
a_2			e	e	e
a_3				e	e
a_4					e

where $A_0 = \{e\}$, $A_1 = \{a_i\}$, $1 \leq i \leq 4$. One obtains $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_i) = 0,25$, $1 \leq i \leq 4$. The join space ${}^1H$ associated is the following one

H	e	a_1	a_2	a_3	a_4
e	e	H	H	H	H
a_1		A_1	A_1	A_1	A_1
a_2			A_1	A_1	A_1
a_3				A_1	A_1
a_4					A_1

hence $\widetilde{\mu}_1(e) = 0,29$ and $\widetilde{\mu}_1(a_i) = 0,23$, $1 \leq i \leq 4$. It follows ${}^rH = {}^1H$, $\forall r \geq 2$.

a$_2$)

H	e	a_1	a_2	a_3	a_4
e	e	a_1	A_2	A_2	A_2
a_1		A_2	e	e	e
a_2			a_1	a_1	a_1
a_3				a_1	a_1
a_4					a_1

with $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2, a_3, a_4\}$. We have $\widetilde{\mu}(e) = \widetilde{\mu}(a_1) = 1$ and $\widetilde{\mu}(a_2) = \widetilde{\mu}(a_3) = \widetilde{\mu}(a_4) = 0,(3)$. Then

${}^1H$	e	a_1	a_2	a_3	a_4
e	e, a_1	e, a_1	H	H	H
a_1		e, a_1	H	H	H
a_2			A_2	A_2	A_2
a_3				A_2	A_2
a_4					A_2

is the join space associated and the membership function is: $\widetilde{\mu}_1(e) = \widetilde{\mu}_1(a_1) = 0,28$, $\widetilde{\mu}_1(a_i) = 0,26$, $2 \leq i \leq 4$. So ${}^rH = {}^1H$, $\forall r \geq 2$.

a$_3$)

H	e	a_1	a_2	a_3	a_4
e	e	A_1	A_1	A_2	A_2
a_1		A_2	A_2	e	e
a_2			A_2	e	e
a_3				A_1	A_1
a_4					A_1

where $A_0 = \{e\}$, $A_1 = \{a_1, a_2\}$, $A_3 = \{a_3, a_4\}$. We calculate $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_i) = 0,5$, $1 \leq i \leq 4$. The join space associated is the same as in a$_1$), so ${}^rH = {}^1H$, $\forall r \geq 1$.

a$_4$)

H	e	a_1	a_2	a_3	a_4
e	e	a_1	a_2	A_3	A_3
a_1		a_2	A_3	e	e
a_2			e	a_1	a_1
a_3				a_2	a_2
a_4					a_2

where $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3, a_4\}$. So, $\widetilde{\mu}(e) = \widetilde{\mu}(a_1) = \widetilde{\mu}(a_2) = 1$, $\widetilde{\mu}(a_3) = \widetilde{\mu}(a_4) = 0,5$; the membership function $\widetilde{\mu}_1$ is the same as in a$_2$) and we have again $s.f.g.(H) = 1$.

a$_5$) For $G = (K, \cdot)$ the Klein's group and $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3, a_4\}$, we obtain

H	e	a_1	a_2	a_3	a_4
e	e	a_1	a_2	A_3	A_3
a_1		e	A_3	a_2	a_2
a_2			e	a_1	a_1
a_3				e	e
a_4					e

with the join space ${}^1H$ equal to the one of a$_4$), hence ${}^rH = {}^1H$, $\forall r \geq 2$.

b) The following complete hypergroups are not 1-hypergroups :

b$_1$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_1	A_1	A_1
a_1		A_0	A_1	A_1	A_1
a_2			A_0	A_0	A_0
a_3				A_0	A_0
a_4					A_0

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2, a_3, a_4\}$ and $\widetilde{\mu}(e) = \widetilde{\mu}(a_1) = 0,5$, $\widetilde{\mu}(a_2) = \widetilde{\mu}(a_3) = \widetilde{\mu}(a_4) = 0,(3)$. The join space ${}^1H$ is the following:

${}^1H$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	H	H	H
a_1		A_0	H	H	H
a_2			A_1	A_1	A_1
a_3				A_1	A_1
a_4					A_1

which leads to ${}^rH = {}^1H$, $\forall r \geq 1$.

b$_2$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_0	A_1	A_1
a_1		A_0	A_0	A_1	A_1
a_2			A_0	A_1	A_1
a_3				A_0	A_0
a_4					A_0

where $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3, a_4\}$. $\widetilde{\mu}(e) = \widetilde{\mu}(a_1) = \widetilde{\mu}(a_2) = 0,(3)$, $\widetilde{\mu}(a_3) = \widetilde{\mu}(a_4) = 0,5$.

The join space associated is isomorphic with the one of the case b$_1$) and therefore $s.f.g.(H) = 1$.

b$_3$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_0	A_0	a_4
a_1		A_0	A_0	A_0	a_4
a_2			A_0	A_0	a_4
a_3				A_0	a_4
a_4					A_0

with $A_0 = \{e, a_1, a_2, a_3\}$, $A_1 = \{a_4\}$. We have $\widetilde{\mu}(e) = \widetilde{\mu}(a_i) = 0,25$, $1 \leq i \leq 3$, $\widetilde{\mu}(a_4) = 1$. The join space associated is

${}^1H$	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_0	A_0	H
a_1		A_0	A_0	A_0	H
a_2			A_0	A_0	H
a_3				A_0	H
a_4					a_4

for which $\widetilde{\mu}_1(e) = \widetilde{\mu}_1(a_i) = 0,233$, $1 \leq i \leq 3$, $\widetilde{\mu}_1(a_4) = 0,289$. Hence ${}^rH = {}^1H$, $\forall r \geq 2$.

b$_4$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	a_2	A_2	A_2
a_1		A_0	a_2	A_2	A_2
a_2			A_2	A_0	A_0
a_3				a_2	a_2
a_4					a_2

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3, a_4\}$. We have $\widetilde{\mu}(e) = \widetilde{\mu}(a_i) = 0,5$, $i \in \{1, 3, 4\}$, $\widetilde{\mu}(a_2) = 1$. It follows the join space

${}^1H$	e	a_1	a_2	a_3	a_4
e	H_2	H_2	H	H_2	H_2
a_1		H_2	H	H_2	H_2
a_2			a_2	H	H
a_3				H_2	H_2
a_4					H_2

where $H_2 = H \setminus \{2\}$, which is isomorphic with the preceding one. Then ${}^rH = {}^1H$, $\forall r \geq 1$.

b$_5$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	A_0	a_3	a_4
a_1		A_0	A_0	a_3	a_4
a_2			A_0	a_3	a_4
a_3				a_4	A_0
a_4					a_3

where $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3\}$, $A_2 = \{a_4\}$. It leads to the join space from the case b$_1$).

b$_6$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	a_2	a_3	a_4
a_1		A_0	a_2	a_3	a_4
a_2			a_3	a_4	A_0
a_3				A_0	a_2
a_4					a_3

with $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3\}$, $A_3 = \{a_4\}$. The join space is the one of b$_1$).

b$_7$)

H	e	a_1	a_2	a_3	a_4
e	A_0	A_0	a_2	a_3	a_4
a_1		A_0	a_2	a_3	a_4
a_2			A_0	a_4	a_3
a_3				A_0	a_2
a_4					A_0

with $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3\}$, $A_3 = \{a_4\}$ and $G = (K, \cdot)$ the Klein's group. Again the join space is the one from b$_1$).

iv) a) The following complete hypergroups

a$_1$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	A_3	A_3	A_3
a_1		a_2	A_3	e	e	e
a_2			e	a_1	a_1	a_1
a_3				a_2	a_2	a_2
a_4					a_2	a_2
a_5						a_2

where $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3, a_4, a_5\}$, $G = (\mathbb{Z}_4, +)$.

a$_2$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	a_3	a_4	a_5
a_1		A_0	A_0	a_3	a_4	a_5
a_2			A_0	a_3	a_4	a_5
a_3				a_4	a_5	A_0
a_4					A_0	a_3
a_5						a_4

where $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3\}$, $A_2 = \{a_4\}$, $A_3 = \{a_5\}$, $G = (\mathbb{Z}_4, +)$.

a$_3$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	A_3	A_3	A_3
a_1		e	A_3	a_2	a_2	a_2
a_2			e	a_1	a_1	a_1
a_3				e	e	e
a_4					e	e
a_5						e

with $G = (K, \cdot)$ the Klein's group, $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3, a_4, a_5\}$.

a$_4$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	a_3	a_4	a_5
a_1		a_0	A_0	a_3	a_4	a_5
a_2			A_0	a_3	a_4	a_5
a_3				A_0	a_5	a_4
a_4					A_0	a_3
a_5						A_0

with $G = (K, \cdot)$ the Klein's group, $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3\}$, $A_2 = \{a_4\}$, $A_3 = \{a_5\}$.

have the property $|H/R| = 2$ and all the equivalence classes have the same cardinality. By the Case I a) of the paragraph 3.3.3, we obtain $s.f.g.(H) = 2$.

b) The other complete hypergroups of order 6 have $s.f.g.(H) = 1$. We list them, without writing the join spaces and the membership functions associated.

b$_1$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	A_1	A_1	A_1	A_1	A_1
a_1		e	e	e	e	e
a_2			e	e	e	e
a_3				e	e	e
a_4					e	e
a_5						e

where
$A_0 = \{e\}$,
$A_1 = \{a_1, a_2, a_3, a_4, a_5\}$.
H is an 1-hypergroup.

b$_2$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_1	A_1	A_1	A_1
a_1		A_0	A_1	A_1	A_1	A_1
a_2			A_0	A_0	A_0	A_0
a_3				A_0	A_0	A_0
a_4					A_0	A_0
a_5						A_0

where
$A_0 = \{e, a_1\}$,
$A_1 = \{a_2, a_3, a_4, a_5\}$.

b$_3$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	A_1	A_1	A_1
a_1		A_0	A_0	A_1	A_1	A_1
a_2			A_0	A_1	A_1	A_1
a_3				A_0	A_0	A_0
a_4					A_0	A_0
a_5						A_0

where $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3, a_4, a_5\}$.

b$_4$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	A_0	A_1	A_1
a_1		A_0	A_0	A_0	A_1	A_1
a_2			A_0	A_0	A_1	A_1
a_3				A_0	A_1	A_1
a_4					A_0	A_0
a_5						A_0

where $A_0 = \{e, a_1, a_2, a_3\}$, $A_1 = \{a_4, a_5\}$.

b$_5$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	A_0	A_0	a_5
a_1		A_0	A_0	A_0	A_0	a_5
a_2			A_0	A_0	A_0	a_5
a_3				A_0	A_0	a_5
a_4					A_0	a_5
a_5						A_0

with $A_0 = \{e, a_1, a_2, a_3, a_4\}$, $A_1 = \{a_5\}$.

b$_6$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	A_2	A_2	A_2	A_2
a_1		A_2	e	e	e	e
a_2			a_1	a_1	a_1	a_1
a_3				a_1	a_1	a_1
a_4					a_1	a_1
a_5						a_1

where $A_0 = \{e\}$, $A_1 = \{a_1, a_2, a_3, a_4\}$, $A_1 = \{a_5\}$. H is an 1-hypergroup.

b$_7$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	A_1	A_1	A_2	A_2	A_2
a_1		A_2	A_2	e	e	e
a_2			A_2	e	e	e
a_3				A_1	A_1	A_1
a_4					A_1	A_1
a_5						A_1

where $A_0 = \{e\}$, $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, a_4, a_5\}$. H is an 1-hypergroup.

b$_8$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	A_2	A_2	A_2
a_1		A_0	a_2	A_2	A_2	A_2
a_2			A_2	A_0	A_0	A_0
a_3				a_2	a_2	a_2
a_4					a_2	a_2
a_5						a_2

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3, a_4, a_5\}$.

b$_9$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_1	A_1	A_2	A_2
a_1		A_0	A_1	A_1	A_2	A_2
a_2			A_2	A_2	A_0	A_0
a_3				A_2	A_0	A_0
a_4					A_1	A_1
a_5						A_1

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2, a_3\}$, $A_2 = \{a_4, a_5\}$.

b$_{10}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	a_3	A_2	A_2
a_1		A_0	A_0	a_3	A_2	A_2
a_2			A_0	a_3	A_2	A_2
a_3				A_2	A_0	A_0
a_4					a_3	a_3
a_5						a_3

where $A_0 = \{e, a_1, a_2\}$, $A_1 = \{a_3\}$, $A_2 = \{a_4, a_5\}$.

b$_{11}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	A_0	A_0	a_4	a_5
a_1		A_0	A_0	A_0	a_4	a_5
a_2			A_0	A_0	a_4	a_5
a_3				A_0	a_4	a_5
a_4					a_5	A_0
a_5						a_4

where $A_0 = \{e, a_1, a_2, a_3\}$, $A_1 = \{a_4\}$, $A_2 = \{a_5\}$.

b$_{12}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	A_2	A_2	A_3	A_3
a_1		A_2	A_3	A_3	e	e
a_2			e	e	a_1	a_1
a_3				e	a_1	a_1
a_4					A_2	A_2
a_5						A_2

with $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2, a_3\}$, $A_3 = \{a_4, a_5\}$. $G = (\mathbb{Z}_4, +)$. H is an 1-hypergroup.

b$_{13}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	a_3	A_3	A_3
a_1		A_0	a_2	a_3	A_3	A_3
a_2			a_3	A_3	A_0	A_0
a_3				A_0	a_2	a_2
a_4					a_3	a_3
a_5						a_3

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3\}$, $A_3 = \{a_4, a_5\}$. $G = (\mathbb{Z}_4, +)$.

b$_{14}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	A_2	A_2	A_3	A_3
a_1		e	A_3	A_3	A_2	A_2
a_2			e	e	a_1	a_1
a_3				e	a_1	a_1
a_4					e	e
a_5						e

where $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2, a_3\}$, $A_3 = \{a_4, a_5\}$. $G = (K, \cdot)$ the Klein's group. H is an 1-hypergroup.

b$_{15}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	a_3	A_3	A_3
a_1		A_0	a_2	a_3	A_3	A_3
a_2			A_0	A_3	a_3	a_3
a_3				A_0	a_2	a_2
a_4					A_0	A_0
a_5						A_0

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3\}$, $A_3 = \{a_4, a_5\}$. $G = (K, \cdot)$ the Klein's group.

b$_{16}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	A_4	A_4
a_1		a_2	a_3	A_4	e	e
a_2			A_4	e	a_1	a_1
a_3				a_1	a_2	a_2
a_4					a_3	a_3
a_5						a_3

where $A_0 = \{e\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3\}$, $A_4 = \{a_4, a_5\}$. H is an 1-hypergroup.

b$_{17}$)

H	e	a_1	a_2	a_3	a_4	a_5
e	A_0	A_0	a_2	a_3	a_4	a_5
a_1		A_0	a_2	a_3	a_4	a_5
a_2			a_3	a_4	a_5	A_0
a_3				a_5	A_0	a_2
a_4					a_2	a_3
a_5						a_4

where $A_0 = \{e, a_1\}$, $A_1 = \{a_2\}$, $A_2 = \{a_3\}$, $A_3 = \{a_4\}$, $A_4 = \{a_5\}$.

Appendix B

The sequence of fuzzy sets and of join spaces determined by all the i.p.s. hypergroups of order less than or equal to 7

Theorem 3.3.26. (see [14], [15])

(i) *There is only one i.p.s. hypergroup H of order 3 and $s.f.g.(H) = 1$.*

(ii) *There are three non isomorphic i.p.s. hypergroups H_1, H_2, H_3 of order 4 and $s.f.g.(H_1) = s.f.g.(H_2), s.f.g.(H_3) = 2$.*

(iii) *There are eight non isomorphic i.p.s. hypergroups of order 5: one of them has $s.f.g.(H) = 2$ and the other one have $s.f.g.(H) = 1$.*

(iv) *There are nineteen non isomorphic i.p.s. hypergroups of order 6: fourteen of them have $s.f.g.(H) = 1$, four of them are of $s.f.g.(H) = 2$ and one is with $s.f.g.(H) = 3$.*

(v) *There are thirty-six non isomorphic i.p.s. hypergroups of order 7: two of them have $f.g.(H) = 1$, twenty-five of them have $s.f.g.(H) = 1$, eight of them are with $s.f.g.(H) = 2$ and only for one of them $s.f.g.(H) = 3$*

Proof. We shall denote, in the following tables, $\forall\, s$, $A_s = H \setminus \{s\}$.

(i) We have

H	0	1	2
0	0	1	2
1		0, 2	1
2			0

Then
$\widetilde{\mu}(0) = \widetilde{\mu}(2) = 0,833$, $\widetilde{\mu}(1) = 1$,
whence the join space associated is

^1H	0	1	2
0	$0,2$	H	$0,2$
1		1	H
2			$0,2$

It follows:
$\widetilde{\mu}_1(0)=\widetilde{\mu}_1(2)=0,417$, $\widetilde{\mu}_1(1)=0,467$.
Then $\forall\, r>1$, ${}^rH={}^1H$.

(ii) Let us suppose H of order 4.

a)

H	0	1	2	3
0	0	1	2	3
1		2	0	3
2			1	3
3				$0,1,2$

We have:
$\widetilde{\mu}(0)=\widetilde{\mu}(1)=\widetilde{\mu}(2)=0,833$
$\widetilde{\mu}(3)=1$.
Therefore the join space associated is

^1H	0	1	2	3
0	A_3	A_3	A_3	H
1		A_3	A_3	H
2			A_3	H
3				3

From this, one obtains:
$\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=\widetilde{\mu}_1(2)=0,3$
$\widetilde{\mu}_1(3)=0,357$, whence
$\forall\, r>1$, ${}^rH={}^1H$.

b)

H	0	1	2	3
0	0	1	2	3
1		$0,2$	1	3
2			0	3
3				$0,1,2$

We have:
$\widetilde{\mu}(0)=0,708=\widetilde{\mu}(2)$
$\widetilde{\mu}(1)=0,867, \widetilde{\mu}(3)=1$.
Therefore the join space associated is

^1H	0	1	2	3
0	$0,2$	A_3	$0,2$	H
1		1	A_3	$1,3$
2			$0,2$	H
3				3

Hence:
$\widetilde{\mu}_1(0)=\widetilde{\mu}_2(2)=0,361$
$\widetilde{\mu}_1(1)=0,394$, $\widetilde{\mu}_1(3)=0,429$.
One obtains $\forall\, r>1$, ${}^rH={}^1H$.

c)

H	0	1	2	3
0	0	1	2	3
1		2	$0,3$	1
2			1	2
3				0

The membership of H is:
$\widetilde{\mu}(0)=\widetilde{\mu}(3)=0,75$,
$\widetilde{\mu}(1)=\widetilde{\mu}(2)=1$.
We find:

^1H	0	1	2	3
0	$0,3$	H	H	$0,3$
1		$1,2$	$1,2$	H
2			$1,2$	H
3				$0,3$

From this we obtain:
$\widetilde{\mu}_1(i)=0,333$, $\forall\, i$.
Hence ${}^2H$ is the total hypergroup of order 4.

(iii) We consider now the i.p.s. hypergroups of order 5.

a$_1$)

H	0	1	2	3	4
0	0	1	2	3	4
1		$0,2$	1	4	3
2			0	3	4
3				$0,2$	1
4					$0,2$

We have :
$\widetilde{\mu}(0) = \widetilde{\mu}(2) = 0,7$,
$\widetilde{\mu}(1) = \widetilde{\mu}(3) = \widetilde{\mu}(4) = 1$,
whence, the join space associated is the following one:

^1H	0	1	2	3	4
0	$0,2$	H	$0,2$	H	H
1		$1,3,4$	H	$1,3,4$	$1,3,4$
2			$0,2$	H	H
3				$1,3,4$	$1,3,4$
4					$1,3,4$

Now:
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(2) = 0,275$
$\widetilde{\mu}_1(1) = \widetilde{\mu}_1(3) = \widetilde{\mu}(4) = 0,254$.
It follows $\forall\, r > 1$, $^rH = {}^1H$.

a$_2$)

H	0	1	2	3	4
0	0	1	2	3	4
1		$0,2$	1	3	4
2			0	3	4
3				$0,1,2,4$	3
4					$0,1,2$

We have
$\widetilde{\mu}(0) = 0,616 = \widetilde{\mu}(2)$,
$\widetilde{\mu}(1) = 0,763$, $\widetilde{\mu}(4) = 0,892$,
$\widetilde{\mu}(3) = 1$, whence

^1H	0	1	2	3	4
0	$0,2$	$0,1,2$	$0,2$	H	$0,1,2,4$
1		1	$0,1,2$	$1,3,4$	$1,4$
2			$0,2$	H	$0,1,2,4$
3				3	$3,4$
4					4

Then $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(2) = 0,320$,
$\widetilde{\mu}_1(1) = 0,341$, $\widetilde{\mu}_1(4) = 0,364$,
$\widetilde{\mu}_1(3) = 0,385$. Therefore $\forall\, r \geq 2$,
we have $^rH = {}^1H$.

a$_3$)

H	0	1	2	3	4
0	0	1	2	3	4
1		$0,2$	1	3	4
2			0	3	4
3				4	$0,1,2$
4					3

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(2) = 0,633$,
$\widetilde{\mu}(1) = 0,777$, $\widetilde{\mu}(3) = \widetilde{\mu}(4) = 1$.
It follows the join space

^1H	0	1	2	3	4
0	$0,2$	$0,1,2$	$0,2$	H	H
1		1	$0,1,2$	$1,3,4$	$1,3,4$
2			$0,2$	H	H
3				$3,4$	$3,4$
4					$3,4$

whence
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) =$
$= \widetilde{\mu}_1(4) = 0,308$, $\widetilde{\mu}_1(1) = 0,309$.
One obtains the second join space

^2H	0	1	2	3	4
0	$0,2,3,4$	H	$0,2,3,4$	$0,2,3,4$	$0,2,3,4$
1		1	H	H	H
2			$0,2,3,4$	$0,2,3,4$	$0,2,3,4$
3				$0,2,3,4$	$0,2,3,4$
4					$0,2,3,4$

Hence $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(2) = \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = 0,217$, $\widetilde{\mu}_2(1) = 0,578$.
Therefore, $\forall\, r > 2$, ${}^rH = {}^2H$ and $s.f.g.(H) = 2$.

a$_4$)

H	0	1	2	3	4
0	0	1	2	3	4
1		$0,2,3$	1	1	4
2			$0,3$	2	4
3				0	4
4					A_4

We have: $\widetilde{\mu}(0) = 0,616$, $\widetilde{\mu}(1) = 0,892$, $\widetilde{\mu}(2) = 0,763$, $\widetilde{\mu}(3) = 0,616$, $\widetilde{\mu}(4) = 1$, whence, the associated join space is the following one

^1H	0	1	2	3	4
0	$0,3$	A_4	$0,2,3$	$0,3$	H
1		1	$1,2$	A_4	$1,4$
2			2	$0,2,3$	$1,2,4$
3				$0,3$	H
4					4

One finds:
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(3) = 0,320$,
$\widetilde{\mu}_1(1) = 0,364$, $\widetilde{\mu}_1(2) = 0,341$,
$\widetilde{\mu}_1(4) = 0,385$.
Therefore, one obtains
$\forall\, r > 1$, ${}^rH_4 = {}^1H_4$.

a$_5$)

H	0	1	2	3	4
0	0	1	2	3	4
1		$0,2,3$	1	1	4
2			3	0	4
3				2	4
4					A_4

It results:
$\widetilde{\mu}(0) = \widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,716$,
$\widetilde{\mu}(1) = 0,892$, $\widetilde{\mu}(4) = 1$.
Therefore we obtain

^1H	0	1	2	3	4
0	$0,2,3$	A_4	$0,2,3$	$0,2,3$	H
1		1	A_4	A_4	$1,4$
2			$0,2,3$	$0,2,3$	H
3				$0,2,3$	H
4					4

From this one finds:
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,271$,
$\widetilde{\mu}_1(1) = 0,313$, $\widetilde{\mu}_1(4) = 0,355$.
Then we have:
$\forall\, r > 1$, ${}^rH = {}^1H$.

a$_6$)

H	0	1	2	3	4
0	0	1	2	3	4
1		A_1	1	1	1
2			3	$0,4$	2
3				2	3
4					0

Hence:
$\widetilde{\mu}(0) = \widetilde{\mu}(4) = 0,650$,
$\widetilde{\mu}(1) = 1$, $\widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,875$.
It follows the join space

^1H	0	1	2	3	4
0	$0,4$	H	A_1	A_1	$0,4$
1		1	$1,2,3$	$1,2,3$	H
2			$2,3$	$2,3$	A_1
3				$2,3$	A_1
4					$0,4$

Then we have
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(4) = 0,266$,
$\widetilde{\mu}_1(1) = 0,348$,
$\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,306$.
Therefore $\forall\, r > 1$, ${}^rH = {}^1H$.

a$_7$)

H	0	1	2	3	4
0	0	1	2	3	4
1		A_1	1	1	1
2			3	0	4
3				2	4
4					$0,2,3$

We find:
$\widetilde{\mu}(0) = \widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,716$,
$\widetilde{\mu}(1) = 1$, $\widetilde{\mu}(4) = 0,892$.
It follows

^1H	0	1	2	3	4
0	$0,2,3$	H	$0,2,3$	$0,2,3$	A_1
1		1	H	H	$1,4$
2			$0,2,3$	$0,2,3$	A_1
3				$0,2,3$	A_1
4					4

From this we obtain
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,271$,
$\widetilde{\mu}_1(1) = 0,355$, $\widetilde{\mu}_1(4) = 0,313$.
Therefore we have
$\forall\, r > 1$, ${}^rH = {}^1H$.

a$_8$)

H	0	1	2	3	4
0	0	1	2	3	4
1		A_1	1	1	1
2			4	0	3
3				4	2
4					0

Hence:
$\widetilde{\mu}(0) = \widetilde{\mu}(2) = \widetilde{\mu}(3) =$
$= \widetilde{\mu}(4) = 0,85$, $\widetilde{\mu}(1)$=1.
We obtain the same join space we got from the join space ^1H associated with H in a$_3$).
Therefore, we have $s.f.g.(H)$=1.

(iv) Now we study the i.p.s. hypergroups of order 6.

b$_1$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			3	4	5	$0,1$
3				5	$0,1$	2
4					2	3
5						4

Calculating the membership function one finds:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,6$,
$\widetilde{\mu}(2) = \widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$.
So the join space associated with H is

^1H	0	1	2	3	4	5
0	$0,1$	$0,1$	H	H	H	H
1		$0,1$	H	H	H	H
2			$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5$
3				$2,3,4,5$	$2,3,4,5$	$2,3,4,5$
4					$2,3,4,5$	$2,3,4,5$
5						$2,3,4,5$

From this one obtains: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,233$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) =$ $= \widetilde{\mu}_1(5) = 0,208$. Therefore $\forall\, r,\ {}^rH = {}^1H$.

b$_2$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	4	3	5
2			3, 4	5	5	0, 1
3				1	0	2
4					1	2
5						3, 4

We have: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(3) =$ $= \widetilde{\mu}(4) = 0,833$, $\widetilde{\mu}(2) = \widetilde{\mu}(5) = 1$. Therefore the join space associated is

${}^1H$	0	1	2	3	4	5
0	0, 1, 3, 4	0, 1, 3, 4	H	0, 1, 3, 4	0, 1, 3, 4	H
1		0, 1, 3, 4	H	0, 1, 3, 4	0, 1, 3, 4	H
2			2, 5	H	H	2, 5
3				0, 1, 3, 4	0, 1, 3, 4	H
4					0, 1, 3, 4	H
5						2, 5

whence: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(3) = 0,208 = \widetilde{\mu}_1(4)$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(5) = 0,233$ from which $\forall\, r,\ {}^rH = {}^1H$.

b$_3$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		2	0	3	4	5
2			1	3	4	5
3				0, 1, 2	5	4
4					0, 1, 2	3
5						0, 1, 2

One finds:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,667$,
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$,
whence, we obtain

${}^1H$	0	1	2	3	4	5
0	0, 1, 2	0, 1, 2	0, 1, 2	H	H	H
1		0, 1, 2	0, 1, 2	H	H	H
2			0, 1, 2	H	H	H
3				3, 4, 5	3, 4, 5	3, 4, 5
4					3, 4, 5	3, 4, 5
5						3, 4, 5

From this we have: $\forall\, u \in H$, $\widetilde{\mu}_1(u) = 0,222$, that is, ${}^2H$ is a total hypergroup of order 6 and $s.f.g.(H) = 2$.

b$_4$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		2	0	3	4	5
2			1	3	4	5
3				5	$0,1,2$	4
4					5	3
5						$0,1,2$

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,667$,
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$,
so the same membership function as for the hypergroup H from b$_3$), whence ${}^2H$ is a total hypergroup and $s.f.g.(H) = 2$.

b$_5$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		2	3	0	4	5
2			0	1	4	5
3				2	4	5
4					$0,1,2,3$	5
5						$0,1,2,3,4$

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) =$
$= \widetilde{\mu}(3) = 0,742$,
$\widetilde{\mu}(4) = 0,911$, $\widetilde{\mu}(5) = 1$,
from this we obtain the join space

${}^1H$	0	1	2	3	4	5
0	$0,1,2,3$	$0,1,2,3$	$0,1,2,3$	$0,1,2,3$	A_5	H
1		$0,1,2,3$	$0,1,2,3$	$0,1,2,3$	A_5	H
2			$0,1,2,3$	$0,1,2,3$	A_5	H
3				$0,1,2,3$	A_5	H
4					4	$4,5$
5						5

Now, the membership function is $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,333$, $\widetilde{\mu}_1(4) = 0,260$, $\widetilde{\mu}_1(5) = 0,303$, so, we find $\forall\, r > 1$, ${}^rH = {}^1H$.

b$_6$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	3	2	4	5
2			0	1	4	5
3				0	4	5
4					$0,1,2,3$	5
5						$0,1,2,3,4$

One finds:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) =$
$= \widetilde{\mu}(3) = 0,742$,
$\widetilde{\mu}(4) = 0,911$, $\widetilde{\mu}(5) = 1$,
that is the same membership function as for the previous H, so we have $\forall\, r > 1$, ${}^1H = {}^rH$.

b$_7$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		2	3	0	4	5
2			0	1	4	5
3				2	4	5
4					5	$0,1,2,3$
5						4

We find:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,75$,
$\widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$.
It follows the following join space

^1H	0	1	2	3	4	5
0	$0,1,2,3$	$0,1,2,3$	$0,1,2,3$	$0,1,2,3$	H	H
1		$0,1,2,3$	$0,1,2,3$	$0,1,2,3$	H	H
2			$0,1,2,3$	$0,1,2,3$	H	H
3				$0,1,2,3$	H	H
4					$4,5$	$4,5$
5						$4,5$

From ^1H one obtains: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,75$, $\widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 1$. Therefore $\forall\, r > 1$, $^rH = {}^1H$.

b$_8$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	3	2	4	5
2			0	1	4	5
3				0	4	5
4					5	$0,1,2,3$
5						4

The corresponding $\widetilde{\mu}$ is as it follows:
$\widetilde{\mu}(0)=\widetilde{\mu}(1)=\widetilde{\mu}(2)=\widetilde{\mu}(3)=0,75$,
$\widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$, whence we obtain the same join space as in previous case. So, $s.f.g.(H) = 1$.

b$_9$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		2	0	3	4	5
2			1	3	4	5
3				4	$0,1,2$	5
4					3	5
5						A_5

The membership function associated with H is:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,533$,
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,9$.
By consequence, we obtain the first join space

^1H	0	1	2	3	4	5
0	$0,1,2$	$0,1,2$	$0,1,2$	A_5	A_5	H
1		$0,1,2$	$0,1,2$	A_5	A_5	H
2			$0,1,2$	A_5	A_5	H
3				$3,4$	$3,4$	$3,4,5$
4					$3,4$	$3,4,5$
5						5

From this we obtain:
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(2) = 0,237$,
$\widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,258$,
$\widetilde{\mu}_1(5) = 0,303$,
whence we get $\forall\, r$, $^rH = {}^1H$.

b$_{10}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		2	0	3	4	5
2			1	3	4	5
3				$0,1,2$	4	5
4					5	$0,1,2,3$
5						4

We find:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,639$,
$\widetilde{\mu}(3) = 0,812$, $\widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$,
whence the associated join space is the following one

^1H	0	1	2	3	4	5
0	$0,1,2$	$0,1,2$	$0,1,2$	$0,1,2,3$	H	H
1		$0,1,2$	$0,1,2$	$0,1,2,3$	H	H
2			$0,1,2$	$0,1,2,3$	H	H
3				3	$3,4,5$	$3,4,5$
4					$4,5$	$4,5$
5						$4,5$

Then we have: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = = \widetilde{\mu}_1(2) = 0,241$, $\widetilde{\mu}_1(3) = 0,254$, $\widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,267$. It follows: $\forall\, r,\ {}^rH = {}^1H$.

b$_{11}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			$0,1$	4	3	5
3				$0,1$	2	5
4					$0,1$	5
5						A_5

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,617$,
$\widetilde{\mu}(2) = \widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,886$,
$\widetilde{\mu}(5) = 1$.
The corresponding join space is

^1H	0	1	2	3	4	5
0	$0,1$	$0,1$	A_5	A_5	A_5	H
1		$0,1$	A_5	A_5	A_5	H
2			$2,3,4$	$2,3,4$	$2,3,4$	$2,3,4,5$
3				$2,3,4$	$2,3,4$	$2,3,4,5$
4					$2,3,4$	$2,3,4,5$
5						5

The associated membership function is: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,253$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,255$, $\widetilde{\mu}_1(5) = 0,288$, whence, we obtain $\forall\, r,\ {}^rH = {}^1H$.

b$_{12}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			4	$0,1$	3	5
3				4	2	5
4					$0,1$	5
5						A_5

One finds that the membership function determined by H coincides with the one determined by the previous hypergroup, therefore we have $\forall\, r,\ {}^rH = {}^1H$.

b$_{13}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			3	$0,1$	4	5
3				2	4	5
4					$0,1,2,3$	5
5						A_5

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,575$,
$\widetilde{\mu}(2) = 0,778 = \widetilde{\mu}(3)$,
$\widetilde{\mu}(4) = 0,911$, $\widetilde{\mu}(5) = 1$.
So, one obtains the following join space

^1H	0	1	2	3	4	5
0	$0,1$	$0,1$	$0,1,2,3$	$0,1,2,3$	A_5	H
1		$0,1$	$0,1,2,3$	$0,1,2,3$	A_5	H
2			$2,3$	$2,3$	$2,3,4$	$2,3,4,5$
3				$2,3$	$2,3,4$	$2,3,4,5$
4					4	$4,5$
5						5

The membership function corresponding to ^1H is $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,273$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,279$, $\widetilde{\mu}_1(4) = 0,305$, $\widetilde{\mu}_1(5) = 0,333$, whence, we obtain $\forall\, r > 1$, $^rH = {}^1H$.

b$_{14}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			3	$0,1$	4	5
3				2	4	5
4					5	$0,1,2,3$
5						4

One finds:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,583$,
$\widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,786$,
$\widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$.
The corresponding join space is

^1H	0	1	2	3	4	5
0	$0,1$	$0,1$	$0,1,2,3$	$0,1,2,3$	H	H
1		$0,1$	$0,1,2,3$	$0,1,2,3$	H	H
2			$2,3$	$2,3$	$2,3,4,5$	$2,3,4,5$
3				$2,3$	$2,3,4,5$	$2,3,4,5$
4					$4,5$	$4,5$
5						$4,5$

^1H determines the function: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,267$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,262$. From this, one obtains the join space

^2H	0	1	2	3	4	5
0	$0,1,4,5$	$0,1,4,5$	H	H	$0,1,4,5$	$0,1,4,5$
1		$0,1,4,5$	H	H	$0,1,4,5$	$0,1,4,5$
2			$2,3$	$2,3$	H	H
3				$2,3$	H	H
4					$0,1,4,5$	$0,1,4,5$
5						$0,1,4,5$

hence: $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = \widetilde{\mu}_2(4) = \widetilde{\mu}_2(5) = 0,208$, $\widetilde{\mu}_2(2) = \widetilde{\mu}_2(3) = 0,233$. It follows $\forall\, r > 2$, $^rH = {}^2H \neq {}^1H$ and $s.f.g.(H) = 2$.

b$_{15}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			$0,1$	3	4	5
3				$0,1,2$	5	4
4					$0,1,2$	3
5						$0,1,2$

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,583$,
$\widetilde{\mu}(2) = 0,714$,
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = 1$.
The associated join space is the following one

^1H	0	1	2	3	4	5
0	$0,1$	$0,1$	$0,1,2$	H	H	H
1		$0,1$	$0,1,2$	H	H	H
2			2	$2,3,4,5$	$2,3,4,5$	$2,3,4,5$
3				$3,4,5$	$3,4,5$	$3,4,5$
4					$3,4,5$	$3,4,5$
5						$3,4,5$

The corresponding membership function is: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,267$, $\widetilde{\mu}_1(2) = 0,254$, $\widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,24$. From this one obtains $\forall\, r \geq 2$, $^rH = {}^1H$.

b$_{16}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			$0,1$	3	4	5
3				5	$0,1,2$	4
4					5	3
5						$0,1,2$

One finds the same membership function as in the previous case, therefore we have $s.f.g.(H) = 1$.

b$_{17}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			$0,1$	3	4	5
3				$0,1,2$	4	5
4					$0,1,2,3$	5
5						A_5

We have:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,547$,
$\widetilde{\mu}(2) = 0,683$, $\widetilde{\mu}(3) = 0,806$,
$\widetilde{\mu}(4) = 0,911$, $\widetilde{\mu}(5) = 1$.
It results the following join space

^1H	0	1	2	3	4	5
0	$0,1$	$0,1$	$0,1,2$	$0,1,2,3$	A_5	H
1		$0,1$	$0,1,2$	$0,1,2,3$	A_5	H
2			2	$2,3$	$2,3,4$	$2,3,4,5$
3				3	$3,4$	$3,4,5$
4					4	$4,5$
5						5

We find $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,29$, $\widetilde{\mu}_1(2) = 0,302$, $\widetilde{\mu}_1(3) = 0,317$, $\widetilde{\mu}_1(4) = 0,332$, $\widetilde{\mu}_1(5) = 0,348$. It follows $\forall\, r > 1$, $^rH = {}^1H$.

b$_{18}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			0, 1	3	4	5
3				4	0, 1, 2, 5	3
4					3	4
5						0, 1, 2

The membership function is
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,556$,
$\widetilde{\mu}(2) = 0,690$,
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = 1$, $\widetilde{\mu}(5) = 0,813$.
The associated join space is

^1H	0	1	2	3	4	5
0	0, 1	0, 1	0, 1, 2	H	H	0, 1, 2, 5
1		0, 1	0, 1, 2	H	H	0, 1, 2, 5
2			2	2, 3, 4, 5	2, 3, 4, 5	2, 5
3				3, 4	3, 4	3, 4, 5
4					3, 4	3, 4, 5
5						5

From this we have: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,283$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(5) = 0,29$. One obtains the following join space

^2H	0	1	2	3	4	5
0	0, 1, 3, 4	0, 1, 3, 4	H	0, 1, 3, 4	0, 1, 3, 4	H
1		0, 1, 3, 4	H	0, 1, 3, 4	0, 1, 3, 4	H
2			2, 5	H	H	2, 5
3				0, 1, 3, 4	0, 1, 3, 4	H
4					0, 1, 3, 4	H
5						2, 5

We have clearly ^2H coincides with the second join space associated with the hypergroup from b$_2$). Therefore $\forall\, r > 2$, $^rH = {}^2H$ and $s.f.g.(H) = 2$.

b$_{19}$)

H	0	1	2	3	4	5
0	0	1	2	3	4	5
1		0	2	3	4	5
2			0, 1	3	4	5
3				4	0, 1, 2	5
4					3	5
5						A_5

One finds:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,561$,
$\widetilde{\mu}(2) = 0,695$,
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,9$,
$\widetilde{\mu}(5) = 1$.
One observes

^1H	0	1	2	3	4	5
0	0, 1	0, 1	0, 1, 2	A_5	A_5	H
1		0, 1	0, 1, 2	A_5	A_5	H
2			2	2, 3, 4	2, 3, 4	2, 3, 4, 5
3				3, 4	3, 4	3, 4, 5
4					3, 4	3, 4, 5
5						5

Hence we find: $\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=0,280$, $\widetilde{\mu}_1(2)=0,2797$, $\widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,2858$,

$\widetilde{\mu}_1(5) = 0,318$. Therefore, we have the second join space

^2H	0	1	2	3	4	5
0	$0,1$	$0,1$	$0,1,2$	$0,1,3,4$	$0,1,3,4$	A_2
1		$0,1$	$0,1,2$	$0,1,3,4$	$0,1,3,4$	A_2
2			2	A_5	A_5	H
3				$3,4$	$3,4$	$3,4,5$
4					$3,4$	$3,4,5$
5						5

From this we have: $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = 0,27948$, $\widetilde{\mu}_2(2) = 0,315 = \widetilde{\mu}_2(5)$. The function $\widetilde{\mu}_2$ determines the third join space

^3H	0	1	2	3	4	5
0	$0,1,3,4$	$0,1,3,4$	H	$0,1,3,4$	$0,1,3,4$	H
1		$0,1,3,4$	H	$0,1,3,4$	$0,1,3,4$	H
2			$2,5$	H	H	$2,5$
3				$0,1,3,4$	$0,1,3,4$	H
4					$0,1,3,4$	H
5						$2,5$

Finally, we obtain $\widetilde{\mu}_3(0)=\widetilde{\mu}_3(1)=\widetilde{\mu}_3(3)=\widetilde{\mu}_3(4)=0,208$, $\widetilde{\mu}_3(2)=\widetilde{\mu}_3(5)=0,233$. Therefore we obtain $\forall\, r > 3$, ${}^rH = {}^3H$ and $s.f.g.(H) = 3$.

(v) Finally we study the fuzzy grade of the i.p.s. hypergroups of order 7.

c_1)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	3	0	4	5	6
2			0	1	4	5	6
3				2	4	5	6
4					5	K	6
5						4	6
6							A_6

where $K = \{0,1,2,3\}$.
We have $\widetilde{\mu}(0) = \widetilde{\mu}(1) =$
$= \widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,666$
$\widetilde{\mu}(4) = \widetilde{\mu}(5) = 0,916$, $\widetilde{\mu}(6) = 1$.
It follows the join space

^1H	0	1	2	3	4	5	6
0	K	K	K	K	A_6	A_6	H
1		K	K	K	A_6	A_6	H
2			K	K	A_6	A_6	H
3				K	A_6	A_6	H
4					$4,5$	$4,5$	$4,5,6$
5						$4,5$	$4,5,6$
6							6

hence
$\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=\widetilde{\mu}_1(2)=\widetilde{\mu}_1(3)=0,195$
$\widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,223$
$\widetilde{\mu}_1(6) = 0,267$.
Then $\forall\, r > 1, {}^1H = {}^rH$.

c_2)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	3	2	4	5	6
2			0	1	4	5	6
3				0	4	5	6
4					5	K	6
5						4	6
6							A_6

In this case the membership function is the same as in c_1), therefore we have again $\forall\, r, {}^rH = {}^1H$.

c_3)

H	0	1	2	3	4	5	6
0					4	5	6
1					4	5	6
2		$\mathbb{K}$			4	5	6
3					4	5	6
4					6	K	5
5						6	4
6							K

where $\mathbb{K} \in \{\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. We have
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,679$.
$\widetilde{\mu}(4) = \widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$.
It follows the following join space

${}^1H$	0	1	2	3	4	5	6
0	K	K	K	K	H	H	H
1		K	K	K	H	H	H
2			K	K	H	H	H
3				K	H	H	H
4					$4,5,6$	$4,5,6$	$4,5,6$
5						$4,5,6$	$4,5,6$
6							$4,5,6$

and we have
$\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=\widetilde{\mu}_1(2)=\widetilde{\mu}_1(3)=0,186$
$\widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) = 0,195$.
Then $\forall\, r$, ${}^rH = {}^1H$.

c_4)

H	0	1	2	3	4	5	6
0					4	5	6
1					4	5	6
2		$\mathbb{K}$			4	5	6
3					4	5	6
4					K	6	5
5						K	4
6							K

where $\mathbb{K} \in \{\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2\}$.
We have $\widetilde{\mu}$ and $\widetilde{\mu}_1$
in the same way as in the case c_3);
by consequence we have
$\forall\, r > 1, {}^1H = {}^rH$.

c_5)

H	0	1	2	3	4	5	6
0					4	5	6
1					4	5	6
2		$\mathbb{K}$			4	5	6
3					4	5	6
4					K	5	6
5						6	$K \cup \{4\}$
6							5

where $\mathbb{K} \in \{\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2\}$.
We have
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,664$
$\widetilde{\mu}(4) = 0,865$
$\widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$.
It follows the join space

^1H	0	1	2	3	4	5	6
0	K	K	K	K	$K\cup\{4\}$	H	H
1		K	K	K	$K\cup\{4\}$	H	H
2			K	K	$K\cup\{4\}$	H	H
3				K	$K\cup\{4\}$	H	H
4					4	$4,5,6$	$4,5,6$
5						$5,6$	$5,6$
6							$5,6$

We have $\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=\widetilde{\mu}_1(2)=\widetilde{\mu}_1(3)=0,197$, $\widetilde{\mu}_1(4)=0,214$, $\widetilde{\mu}_1(5)=\widetilde{\mu}_1(6)=0,234$.
It follows: ${}^rH={}^1H$, $\forall\, r>1$.

c$_6$)

H	0	1	2	3	4	5	6
0					4	5	6
1					4	5	6
2		$\mathbb{K}$			4	5	6
3					4	5	6
4					K	5	6
5						$K\cup\{4\}$	6
6							A_6

where $\mathbb{K}\in\{\mathbb{Z}_4,\mathbb{Z}_2\times\mathbb{Z}_2\}$.
We have
$\widetilde{\mu}(0)=\widetilde{\mu}(1)=\widetilde{\mu}(2)=\widetilde{\mu}(3)=0,66$
$\widetilde{\mu}(4)=0,862$, $\widetilde{\mu}(5)=0,924$,
$\widetilde{\mu}(6)=1$.
So, we obtain the join space

^1H	0	1	2	3	4	5	6
0	K	K	K	K	$K\cup\{4\}$	A_6	H
1		K	K	K	$K\cup\{4\}$	A_6	H
2			K	K	$K\cup\{4\}$	A_6	H
3				K	$K\cup\{4\}$	A_6	H
4					4	$4,5$	$4,5,6$
5						5	$5,6$
6							6

We have $\widetilde{\mu}_1(0)=0,202$, $\widetilde{\mu}_1(4)=0,233$, $\widetilde{\mu}_1(5)=0,267$ and $\widetilde{\mu}_1(6)=0,293$
It follows $\forall\, r\geq 2$, ${}^1H={}^rH$

c$_7$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	0	3	4	5	6
2			1	3	4	5	6
3				6	$0,1,2$	4	5
4					5	6	3
5						3	$0,1,2$
6							4

We have
$\widetilde{\mu}(0)=0,524=\widetilde{\mu}(1)=\widetilde{\mu}(2)$
$\widetilde{\mu}(3)=\widetilde{\mu}(4)=\widetilde{\mu}(5)=\widetilde{\mu}(6)=1$
Therefore the join space
associated is the following one

^1H	0	1	2	3	4	5	6
0	$0,1,2$	$0,1,2$	$0,1,2$	H	H	H	H
1		$0,1,2$	$0,1,2$	H	H	H	H
2			$0,1,2$	H	H	H	H
3				$3,4,5,6$	$3,4,5,6$	$3,4,5,6$	$3,4,5,6$
4					$3,4,5,6$	$3,4,5,6$	$3,4,5,6$
5						$3,4,5,6$	$3,4,5,6$
6							$3,4,5,6$

We clearly have:
$\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(2) = 0,195$, $\widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) = 0,186$,
whence $\forall\, r \geq 2$, ${}^rH = {}^1H$.

c$_8$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	0	3	4	5	6
2			1	3	4	5	6
3				4	$0,1,2$	5	6
4					3	5	6
5						6	$0,1,2,3,4$
6							5

We have
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,581$
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,822$
$\widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$
It follows that

^1H	0	1	2	3	4	5	6
0	$0,1,2$	$0,1,2$	$0,1,2$	$0,1,2,3,4$	$0,1,2,3,4$	H	H
1		$0,1,2$	$0,1,2$	$0,1,2,3,4$	$0,1,2,3,4$	H	H
2			$0,1,2$	$0,1,2,3,4$	$0,1,2,3,4$	H	H
3				$3,4$	$3,4$	$3,4,5,6$	$3,4,5,6$
4					$3,4$	$3,4,5,6$	$3,4,5,6$
5						$5,6$	$5,6$
6							$5,6$

Hence: $\widetilde{\mu}_1(0) = 0,216$, $\widetilde{\mu}_1(3) = 0,225$, $\widetilde{\mu}_1(5) = 0,238$.
It follows $\forall\, r \geq 2$, ${}^rH = {}^1H$.

c$_9$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	0	3	4	5	6
2			1	3	4	5	6
3				4	$0,1,2$	5	6
4					3	5	6
5						$0,1,2,3,4$	6
6							A_6

We have
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,576$
$\widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,819$
$\widetilde{\mu}(5) = 0,924$, $\widetilde{\mu}(6) = 1$
We obtain that the join space ${}^1H$ is the following one

^1H	0	1	2	3	4	5	6
0	$0,1,2$	$0,1,2$	$0,1,2$	$0,1,2,3,4$	$0,1,2,3,4$	A_6	H
1		$0,1,2$	$0,1,2$	$0,1,2,3,4$	$0,1,2,3,4$	A_6	H
2			$0,1,2$	$0,1,2,3,4$	$0,1,2,3,4$	A_6	H
3				$3,4$	$3,4$	$3,4,5$	$3,4,5,6$
4					$3,4$	$3,4,5$	$3,4,5,6$
5						5	$5,6$
6							6

We have $\widetilde{\mu}_1(0) = 0,220$, $\widetilde{\mu}_1(3) = 0,239$, $\widetilde{\mu}_1(5) = 0,269$, $\widetilde{\mu}_1(6) = 0,297$. Therefore $\forall\, r \geq 1$, ${}^rH = {}^1H$.

c_{10})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	0	3	4	5	6
2			1	3	4	5	6
3				$0,1,2$	4	5	6
4					5	$0,1,2,3$	6
5						4	6
6							A_6

We have
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,571$,
$\widetilde{\mu}(3) = 0,741$,
$\widetilde{\mu}(4) = \widetilde{\mu}(5) = 0,917$,
$\widetilde{\mu}(6) = 1$.
One obtains

^1H	0	1	2	3	4	5	6
0	$0,1,2$	$0,1,2$	$0,1,2$	$0,1,2,3$	A_6	A_6	H
1		$0,1,2$	$0,1,2$	$0,1,2,3$	A_6	A_6	H
2			$0,1,2$	$0,1,2,3$	A_6	A_6	H
3				3	$3,4,5$	$3,4,5$	$3,4,5,6$
4					$4,5$	$4,5$	$4,5,6$
5						$4,5$	$4,5,6$
6							6

It follows $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(2) = 0,223$, $\widetilde{\mu}_1(3) = 0,232$, $\widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,251$, $\widetilde{\mu}_1(6) = 0,284$. Therefore $\forall\, r > 1$, ${}^rH = {}^1H$.

c_{11})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	0	3	4	5	6
2			1	3	4	5	6
3				$0,1,2$	4	5	6
4					$0,1,2,3$	5	6
5						$0,1,2,3,4$	5
6							A_6

We have
$\widetilde{\mu}(0)=\widetilde{\mu}(1)=\widetilde{\mu}(2)=0,564$,
$\widetilde{\mu}(3)=0,735$, $\widetilde{\mu}(4)=0,837$,
$\widetilde{\mu}(5)=0,936$, $\widetilde{\mu}(6)=1$.
By consequence the join space associated is

^1H	0	1	2	3	4	5	6
0	$0,1,2$	$0,1,2$	$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	A_6	H
1		$0,1,2$	$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	A_6	H
2			$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	A_6	H
3				3	$3,4$	$3,4,5$	$3,4,5,6$
4					4	$4,5$	$4,5,6$
5						5	$5,6$
6							6

It follows $\widetilde{\mu}_1(0) = \widetilde{\mu}(1)_1 = \widetilde{\mu}_1(2) = 0,229$, $\widetilde{\mu}_1(3) = 0,249$, $\widetilde{\mu}_1(4) = 0,272$, $\widetilde{\mu}_1(5) = 0,291$, $\widetilde{\mu}_1(6) = 0,310$, whence we have $\forall\, r \geq 1$, ${}^rH = {}^1H$.

c$_{12}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	0	3	4	5	6
2			1	3	4	5	6
3				$0,1,2$	4	5	6
4					$0,1,2,3$	6	5
5						$0,1,2,3$	4
6							$0,1,2,3$

We have $\widetilde{\mu}(0) = \widetilde{\mu}(1) = \widetilde{\mu}(2) = 0,583$, $\widetilde{\mu}(3) = 0,75$, $\widetilde{\mu}(4) = \widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$. Therefore the join space associated is

^1H	0	1	2	3	4	5	6
0	$0,1,2$	$0,1,2$	$0,1,2$	$0,1,2,3$	H	H	H
1		$0,1,2$	$0,1,2$	$0,1,2,3$	H	H	H
2			$0,1,2$	$0,1,2,3$	H	H	H
3				3	$3,4,5,6$	$3,4,5,6$	$3,4,5,6$
4					$4,5,6$	$4,5,6$	$4,5,6$
5						$4,5,6$	$4,5,6$
6							$4,5,6$

We have $\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=\widetilde{\mu}_1(2)=\widetilde{\mu}_1(4)=\widetilde{\mu}_1(5)=\widetilde{\mu}_1(6)=0,214$, $\widetilde{\mu}_1(3)=0,212$. Therefore the second join space associated is

^2H	0	1	2	3	4	5	6
0	A_3	A_3	A_3	H	A_3	A_3	A_3
1		A_3	A_3	H	A_3	A_3	A_3
2			A_3	H	A_3	A_3	A_3
3				3	H	H	H
4					A_3	A_3	A_3
5						A_3	A_3
6							A_3

It follows $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = \widetilde{\mu}_2(2) = \widetilde{\mu}_2(4) = \widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) \neq \widetilde{\mu}_2(3)$. By consequence $\forall\, r > 2$, ${}^rH = {}^2H$ and $s.f.g.(H) = 2$.

c$_{13}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			0, 1	3	4	5	6
3				0, 1, 2	4	5	6
4					0, 1, 2, 3	6	5
5						0, 1, 2, 3	4
6							0, 1, 2, 3

We have: $\widetilde{\mu}(0)=\widetilde{\mu}(1)=0,512$, $\widetilde{\mu}(2)=0,635$ $\widetilde{\mu}(3)=0,75$, $\widetilde{\mu}(4)=\widetilde{\mu}(5)=\widetilde{\mu}(6)=1$. The corresponding join space is

^1H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2	0, 1, 2, 3	H	H	H
1		0, 1	0, 1, 2	0, 1, 2, 3	H	H	H
2			2	2, 3	2, 3, 4, 5, 6	2, 3, 4, 5, 6	2, 3, 4, 5, 6
3				3	3, 4, 5, 6	3, 4, 5, 6	3, 4, 5, 6
4					4, 5, 6	4, 5, 6	4, 5, 6
5						4, 5, 6	4, 5, 6
6							4, 5, 6

One obtains $\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=0,252$, $\widetilde{\mu}_1(2)=0,250$, $\widetilde{\mu}_1(3)=0,239$, $\widetilde{\mu}_1(4)=\widetilde{\mu}_1(5)=\widetilde{\mu}_1(6)=0,225$. So it's clear that $\forall\, r>1$, $^rH={}^1H$.

c$_{14}$) One can see that one obtains the same join space as in the previous case starting from the following i.p.s. hypergroup

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			0, 1	3	4	5	6
3				0, 1, 2	4	5	6
4					0, 1, 2, 3	6	5
5						4	0, 1, 2, 3
6							4

c$_{15}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			0, 1	3	4	5	6
3				0, 1, 2	4	5	6
4					5	0, 1, 2, 3	6
5						4	6
6							A_6

One finds:
$\widetilde{\mu}(0)=\widetilde{\mu}(1)=0,5$,
$\widetilde{\mu}(2)=0,625$,
$\widetilde{\mu}(3)=0,741$,
$\widetilde{\mu}(4)=\widetilde{\mu}(5)=0,917$,
$\widetilde{\mu}(6)=1$.

By consequence, we obtain the join space

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2$	$0,1,2,3$	A_6	A_6	H
1		$0,1$	$0,1,2$	$0,1,2,3$	A_6	A_6	H
2			2	$2,3$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5,6$
3				3	$3,4,5$	$3,4,5$	$3,4,5,6$
4					$4,5$	$4,5$	$4,5,6$
5						$4,5$	$4,5,6$
6							6

Therefore: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,260$, $\widetilde{\mu}_1(2) = 0,263$, $\widetilde{\mu}_1(3) = 0,262$, $\widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,265$, $\widetilde{\mu}_1(6) = 0,293$.

One obtains that the second join space associated to H is isomorphic to the first join space associated to H.

By consequence $\forall\, r > 1$, ${}^rH \sim {}^1H$, ${}^{2k}H = {}^2H$, ${}^1H = {}^{2k+1}H$ and $f.g.(H) = 1$.

c_{16})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			$0,1$	3	4	5	6
3				$0,1,2$	4	5	6
4					$0,1,2,3$	5	6
5						6	$0,1,2,3,4$
6							5

We have: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,498$, $\widetilde{\mu}(2) = 0,623$, $\widetilde{\mu}(3) = 0,739$, $\widetilde{\mu}(4) = 0,84$, $\widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$, whence the associated join space is

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	H	H
1		$0,1$	$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	H	H
2			2	$2,3$	$2,3,4$	$2,3,4,5,6$	$2,3,4,5,6$
3				3	$3,4$	$3,4,5,6$	$3,4,5,6$
4					4	$4,5,6$	$4,5,6$
5						$5,6$	$5,6$
6							$5,6$

whence: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) = 0,262$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(4) = 0,267$, $\widetilde{\mu}_1(3) = 0,271$. It follows the second associated join space

^2H	0	1	2	3	4	5	6
0	$0,1,5,6$	$0,1,5,6$	A_3	H	A_3	$0,1,5,6$	$0,1,5,6$
1		$0,1,5,6$	A_3	H	A_3	$0,1,5,6$	$0,1,5,6$
2			$2,4$	$2,3,4$	$2,4$	A_3	A_3
3				3	$2,3,4$	H	H
4					$2,4$	A_3	A_3
5						$0,1,5,6$	$0,1,5,6$
6							$0,1,5,6$

Then we have: $\widetilde{\mu}_2(0)$=$\widetilde{\mu}_2(1)$=$\widetilde{\mu}_2(5)$=$\widetilde{\mu}_2(6)$=$0,195$, $\widetilde{\mu}_2(2)$=$\widetilde{\mu}_2(4)$=$0,223$, $\widetilde{\mu}_2(3)$=$0,267$. Therefore $\forall\, r > 2$, ${}^rH = {}^2H$ and $s.f.g.(H) = 2$.

c$_{17}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			$0,1$	3	4	5	6
3				$0,1,2$	4	5	6
4					$0,1,2,3$	5	6
5						$0,1,2,3,4$	6
6							A_6

We have: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,493$, $\widetilde{\mu}(2) = 0,619$, $\widetilde{\mu}(3) = 0,735$, $\widetilde{\mu}(4) = 0,837$, $\widetilde{\mu}(5) = 0,924$, $\widetilde{\mu}(6)$=1.

The join space associated with the former $\widetilde{\mu}$ is the following one

${}^1H$	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	A_6	H
1		$0,1$	$0,1,2$	$0,1,2,3$	$0,1,2,3,4$	A_6	H
2			2	$2,3$	$2,3,4$	$2,3,4,5$	$2,3,4,5,6$
3				3	$3,4$	$3,4,5$	$3,4,5,6$
4					4	$4,5$	$4,5,6$
5						5	$5,6$
6							6

The corresponding membership function is: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,265$, $\widetilde{\mu}_1(2) = 0,274$, $\widetilde{\mu}_1(3) = 0,283$, $\widetilde{\mu}_1(4) = 0,291$, $\widetilde{\mu}_1(5) = 0,303$, $\widetilde{\mu}_1(6) = 0,318$. From this one obtains: $\forall\, r > 1$, ${}^rH = {}^1H$.

c$_{18}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			$0,1$	3	4	5	6
3				$0,1$	4	5	6
4					$0,1$	5	6
5						$0,1,2,3,4$	6
6							A_6

The corresponding $\widetilde{\mu}$ is: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,552$, $\widetilde{\mu}(2) = 0,728$, $\widetilde{\mu}(3) = 0,796$, $\widetilde{\mu}(4) = 0,837$, $\widetilde{\mu}(5) = 0,924$, $\widetilde{\mu}(6) = 1$. Therefore we have the same join space as in the previous case. So, $s.f.g.(H) = 1$.

$c_{19})$

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			0, 1	3	4	5	6
3				4	0, 1, 2	5	6
4					3	5	6
5						6	0, 1, 2, 3, 4
6							5

We obtain $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,510$, $\widetilde{\mu}(2) = 0,633$, $\widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,822$, $\widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$. Then the associated join space is the hypergroup

^1H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2	0, 1, 2, 3, 4	0, 1, 2, 3, 4	H	H
1		0, 1	0, 1, 2	0, 1, 2, 3, 4	0, 1, 2, 3, 4	H	H
2			2	2, 3, 4	2, 3, 4	2, 3, 4, 5, 6	2, 3, 4, 5, 6
3				3, 4	3, 4	3, 4, 5, 6	3, 4, 5, 6
4					3, 4	3, 4, 5, 6	3, 4, 5, 6
5						5, 6	5, 6
6							5, 6

One obtains the membership function: $\widetilde{\mu}_1(0)$=$\widetilde{\mu}_1(1)$=$0,233$, $\widetilde{\mu}_1(2)$=$0,249$, $\widetilde{\mu}_1(3)$=$\widetilde{\mu}_1(4)$=$0,247$, $\widetilde{\mu}_1(5)$=$\widetilde{\mu}_1(6)$=$0,238$. Therefore $\forall\, r > 1$, $^rH = {^1H}$.

$c_{20})$

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			0, 1	3	4	5	6
3				4	0, 1, 2	5	6
4					3	5	6
5						0, 1, 2, 3, 4	6
6							A_6

The corresponding membership function is $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,505$, $\widetilde{\mu}(2) = 0,629$, $\widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,819$, $\widetilde{\mu}(5) = 0,924$, $\widetilde{\mu}(6) = 1$. The associated join space is

^1H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2	0, 1, 2, 3, 4	0, 1, 2, 3, 4	A_6	H
1		0, 1	0, 1, 2	0, 1, 2, 3, 4	0, 1, 2, 3, 4	A_6	H
2			2	2, 3, 4	2, 3, 4	2, 3, 4, 5	2, 3, 4, 5, 6
3				3, 4	3, 4	3, 4, 5	3, 4, 5, 6
4					3, 4	3, 4, 5	3, 4, 5, 6
5						5	5, 6
6							6

This determines the following membership function $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,257$,

$\widetilde{\mu}_1(2) = 0,255$, $\widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,261$, $\widetilde{\mu}_1(5) = 0,281$, $\widetilde{\mu}_1(6) = 0,305$ from which one obtains the second join space:

^2H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2	0, 1, 3, 4	0, 1, 3, 4	0, 1, 3, 4, 5	A_2
1		0, 1	0, 1, 2	0, 1, 3, 4	0, 1, 3, 4	0, 1, 3, 4, 5	A_2
2			2	0, 1, 2, 3, 4	0, 1, 2, 3, 4	A_6	H
3				3, 4	3, 4	3, 4, 5	3, 4, 5, 6
4					3, 4	3, 4, 5	3, 4, 5, 6
5						5	5, 6
6							6

The corresponding membership function is: $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = 0,257$, $\widetilde{\mu}_2(2) = 0,289$, $\widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = 0,256$, $\widetilde{\mu}_2(5) = 0,279$, $\widetilde{\mu}_2(6) = 0,304$. The third associated join space is

^3H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2, 5	0, 1, 3, 4	0, 1, 3, 4	0, 1, 5	0, 1, 2, 5, 6
1		0, 1	0, 1, 2, 5	0, 1, 3, 4	0, 1, 3, 4	0, 1, 5	0, 1, 2, 5, 6
2			2	A_6	A_6	2, 5	2, 6
3				3, 4	3, 4	0, 1, 3, 4, 5	H
4					3, 4	0, 1, 3, 4, 5	H
5						5	2, 5, 6
6							6

Hence, one obtains: $\widetilde{\mu}_3(0)$=$\widetilde{\mu}_3(1)$=$0,255$, $\widetilde{\mu}_3(2)$=$0,292$, $\widetilde{\mu}_3(3)$=$\widetilde{\mu}_3(4)$=$0,252$, $\widetilde{\mu}_3(5)$=$0,270$, $\widetilde{\mu}_3(6)$=$0,311$.
Therefore, $\forall\, r > 3$, $^rH = {}^3H$ and $s.f.g.(H) = 3$.

c$_{21}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			0, 1	3	4	5	6
3				5	0, 1, 2	4	6
4					5	3	6
5						0, 1, 2	6
6							A_6

The associated membership function is:
$\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,524$, $\widetilde{\mu}(2) = 0,646$, $\widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = 0,907$, $\widetilde{\mu}(6) = 1$.
Here is the corresponding join space

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2$	A_6	A_6	A_6	H
1		$0,1$	$0,1,2$	A_6	A_6	A_6	H
2			2	$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5,6$
3				$3,4,5$	$3,4,5$	$3,4,5$	$3,4,5,6$
4					$3,4,5$	$3,4,5$	$3,4,5,6$
5						$3,4,5$	$3,4,5,6$
6							6

One finds: $\widetilde{\mu}_1(0)$=$\widetilde{\mu}_1(1)$=$0,246$, $\widetilde{\mu}_1(2)$=$0,235$, $\widetilde{\mu}_1(3)$=$\widetilde{\mu}_1(4)$=$\widetilde{\mu}_1(5)$=$0,230$, $\widetilde{\mu}_1(6)$=$0,267$, whence one obtains:

^2H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2$	A_6	A_6	A_6	$0,1,6$
1		$0,1$	$0,1,2$	A_6	A_6	A_6	$0,1,6$
2			2	$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$0,1,2,6$
3				$3,4,5$	$3,4,5$	$3,4,5$	H
4					$3,4,5$	$3,4,5$	H
5						$3,4,5$	H
6							6

The corresponding membership function is as follows: $\widetilde{\mu}_2(0)$=$\widetilde{\mu}_2(1)$=$0,251$, $\widetilde{\mu}_2(2) = 0,232$, $\widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = \widetilde{\mu}_2(5) = 0,223$, $\widetilde{\mu}_2(6) = 0,284$. It results that $\forall\, r > 2$, ${}^rH = {}^2H$ and therefore $s.f.g.(H) = 2$.

c$_{22}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			$0,1$	3	4	5	6
3				$0,1,2$	5	4	6
4					$0,1,2$	3	6
5						$0,1,2$	6
6							A_6

We obtain the same membership function as for the previous i.p.s. hypergroup and by consequence the same associated join spaces. Therefore we have: $s.f.g.(H) = 2$.

c$_{23}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			$0,1$	3	4	5	6
3				5	$0,1,2$	6	4
4					6	3	5
5						4	$0,1,2$
6							6

We have: $\widetilde{\mu}(0)=\widetilde{\mu}(1)=0,548$, $\widetilde{\mu}(2)=0,667$, $\widetilde{\mu}(3)=\widetilde{\mu}(4)=\widetilde{\mu}(5)=\widetilde{\mu}(6)=1$.
So, the associated join space has the form

^1H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2	H	H	H	H
1		0, 1	0, 1, 2	H	H	H	H
2			2	2, 3, 4, 5, 6	2, 3, 4, 5, 6	2, 3, 4, 5, 6	2, 3, 4, 5, 6
3				3, 4, 5, 6	3, 4, 5, 6	3, 4, 5, 6	3, 4, 5, 6
4					3, 4, 5, 6	3, 4, 5, 6	3, 4, 5, 6
5						3, 4, 5, 6	3, 4, 5, 6
6							3, 4, 5, 6

It follows $\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=0,234$, $\widetilde{\mu}_1(2)=0,214$, $\widetilde{\mu}_1(3)=\widetilde{\mu}_1(4)=\widetilde{\mu}_1(5)=\widetilde{\mu}_1(6)=0,197$.
Therefore, we have $\forall\, r > 1$, ${}^rH = {}^1H$.

c_{24})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			3	0, 1	4	5	6
3				2	4	5	6
4					0, 1, 2, 3	6	5
5						0, 1, 2, 3	4
6							0, 1, 2, 3

We have: $\widetilde{\mu}(0)=\widetilde{\mu}(1)=0,536$, $\widetilde{\mu}(2)=\widetilde{\mu}(3)=0,719$, $\widetilde{\mu}(4)=\widetilde{\mu}(5)=\widetilde{\mu}(6)=1$.
So, the associated join space is

^1H	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2, 3	0, 1, 2, 3	H	H	H
1		0, 1	0, 1, 2, 3	0, 1, 2, 3	H	H	H
2			2, 3	2, 3	2, 3, 4, 5, 6	2, 3, 4, 5, 6	2, 3, 4, 5, 6
3				2, 3	2, 3, 4, 5, 6	2, 3, 4, 5, 6	2, 3, 4, 5, 6
4					4, 5, 6	4, 5, 6	4, 5, 6
5						4, 5, 6	4, 5, 6
6							4, 5, 6

We calculate: $\widetilde{\mu}_1(0)=\widetilde{\mu}_1(1)=0,238$, $\widetilde{\mu}_1(2)=\widetilde{\mu}_1(3)=0,225$, $\widetilde{\mu}_1(4)=\widetilde{\mu}_1(5)=\widetilde{\mu}_1(6)=0,216$. It results clearly $\forall\, r > 1$, ${}^rH = {}^1H$.

c_{25})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			3	0, 1	4	5	6
3				2	4	5	6
4					5	6	0, 1, 2, 3
5						0, 1, 2, 3	4
6							5

Since $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,536$, $\widetilde{\mu}(2) = \widetilde{\mu}(3) = 0,719$, $\widetilde{\mu}(4) = \widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$, we conclude $\forall\, r > 1$, ${}^rH = {}^1H$ as in the previous case.

c_{26})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			3	0, 1	4	5	6
3				2	4	5	6
4					0, 1, 2, 3	5	6
5						0, 1, 2, 3, 4	6
6							A_6

The corresponding membership function is:
$\widetilde{\mu}(0)=\widetilde{\mu}(1)=0,517$, $\widetilde{\mu}(2)=\widetilde{\mu}(3)=0,702$, $\widetilde{\mu}(4)=0,837$, $\widetilde{\mu}(5)=0,924$, $\widetilde{\mu}(6)=1$.
The function $\widetilde{\mu}$ determines the join space

${}^1H$	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2, 3	0, 1, 2, 3	0, 1, 2, 3, 4	A_6	H
1		0, 1	0, 1, 2, 3	0, 1, 2, 3	0, 1, 2, 3, 4	A_6	H
2			2, 3	2, 3	2, 3, 4	2, 3, 4, 5	2, 3, 4, 5, 6
3				2, 3	2, 3, 4	2, 3, 4, 5	2, 3, 4, 5, 6
4					4	4, 5	4, 5, 6
5						5	5, 6
6							6

From this, one obtains: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,252$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,255$, $\widetilde{\mu}(4) = 0,270$, $\widetilde{\mu}(5) = 0,292$, $\widetilde{\mu}(6) = 0,311$. It follows $\forall\, r > 1$, ${}^rH = {}^1H$.

c_{27})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			3	0, 1	4	5	6
3				2	4	5	6
4					0, 1, 2, 3	5	6
5						6	0, 1, 2, 3, 4
6							5

The membership function associated with this hypergroup is the following:
$\widetilde{\mu}(0)=\widetilde{\mu}(1) = 0,521$, $\widetilde{\mu}(2)=\widetilde{\mu}(3) = 0,706$, $\widetilde{\mu}(4)=0,84$, $\widetilde{\mu}(5)=\widetilde{\mu}(6)=1$.
One obtains the join space

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2,3$	$0,1,2,3$	$0,1,2,3,4$	H	H
1		$0,1$	$0,1,2,3$	$0,1,2,3$	$0,1,2,3,4$	H	H
2			$2,3$	$2,3$	$2,3,4$	$2,3,4,5,6$	$2,3,4,5,6$
3				$2,3$	$2,3,4$	$2,3,4,5,6$	$2,3,4,5,6$
4					4	$4,5,6$	$4,5,6$
5						$5,6$	$5,6$
6							$5,6$

So we have: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,248$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = 0,247$, $\widetilde{\mu}_1(4) = 0,249$, $\widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) = 0,253$. From this, one gets the second join space associated with H

^2H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2,3$	$0,1,2,3$	$0,1,4$	$0,1,4,5,6$	$0,1,4,5,6$
1		$0,1$	$0,1,2,3$	$0,1,2,3$	$0,1,4$	$0,1,4,5,6$	$0,1,4,5,6$
2			$2,3$	$2,3$	$0,1,2,3,4$	H	H
3				$2,3$	$0,1,2,3,4$	H	H
4					4	$4,5,6$	$4,5,6$
5						$5,6$	$5,6$
6							$5,6$

We clearly have: $^2H \simeq {}^1H$ and moreover $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = 0,247$, $\widetilde{\mu}_2(2) = \widetilde{\mu}_2(3) = 0,248$, $\widetilde{\mu}_2(4) = 0,249$, $\widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) = 0,253$, that is $\widetilde{\mu}_2 = \widetilde{\mu}$, by consequence $\forall\, k \geq 1$, $\widetilde{\mu} = \widetilde{\mu}_{2k}$ and $\widetilde{\mu}_1 = \widetilde{\mu}_{2k+1}$. Finally, $\forall\, k \geq 1$, $^{2k}H = {}^2H \sim {}^1H = {}^{2k+1}H$ and therefore $f.g.(H) = 1$.

c_{28})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			$0,1$	4	3	5	6
3				$0,1$	2	5	6
4					$0,1$	5	6
5						6	$0,1,2,3,4$
6							5

We have: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,557$, $\widetilde{\mu}(2) = \widetilde{\mu}(3) = \widetilde{\mu}(4) = 0,8$, $\widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$. Hence one obtains

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	$0,1,2,3,4$	$0,1,2,3,4$	$0,1,2,3,4$	H	H
1		$0,1$	$0,1,2,3,4$	$0,1,2,3,4$	$0,1,2,3,4$	H	H
2			$2,3,4$	$2,3,4$	$2,3,4$	$2,3,4,5,6$	$2,3,4,5,6$
3				$2,3,4$	$2,3,4$	$2,3,4,5,6$	$2,3,4,5,6$
4					$2,3,4$	$2,3,4,5,6$	$2,3,4,5,6$
5						$5,6$	$5,6$
6							$5,6$

The corresponding membership function is : $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = \widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) = 0,231$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,218$. We form the second associated join space

^2H	0	1	2	3	4	5	6
0	$0,1,5,6$	$0,1,5,6$	H	H	H	$0,1,5,6$	$0,1,5,6$
1		$0,1,5,6$	H	H	H	$0,1,5,6$	$0,1,5,6$
2			$2,3,4$	$2,3,4$	$2,3,4$	H	H
3				$2,3,4$	$2,3,4$	H	H
4					$2,3,4$	H	H
5						$0,1,5,6$	$0,1,5,6$
6							$0,1,5,6$

From this we find a function $\widetilde{\mu}_2$ which gives the same join space. Indeed: $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = \widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) = 0,186$, $\widetilde{\mu}_2(2) = \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = 0,195$. Therefore we have $\forall\, r > 2$, $^rH = {}^2H$ and $s.f.g.(H) = 2$.

$c_{29})$

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			4	$0,1$	3	5	6
3				4	2	5	6
4					$0,1$	5	6
5						6	$0,1,2,3,4$
6							5

One find the same membership function $\widetilde{\mu}$ associated with the previous i.p.s. hypergroup, and therefore an equal sequence of join spaces and $s.f.g.(H)$=2.

$c_{30})$

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			4	$0,1$	5	3	6
3				5	2	4	6
4					3	$0,1$	6
5						2	6
6							A_6

We have: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,595$, $\widetilde{\mu}(2) = \widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = 0,896$, $\widetilde{\mu}(6) = 1$, whence the associated join space is

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	A_6	A_6	A_6	A_6	H
1		$0,1$	A_6	A_6	A_6	A_6	H
2			$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5,6$
3				$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5,6$
4					$2,3,4,5$	$2,3,4,5$	$2,3,4,5,6$
5						$2,3,4,5$	$2,3,4,5,6$
6							6

The membership function associated with ^1H is $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,218$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = \widetilde{\mu}_1(5) = 0,201$, $\widetilde{\mu}_1(6) = 0,244$, whence the associated join space is as follows:

^2H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	A_6	A_6	A_6	A_6	$0,1,6$
1		$0,1$	A_6	A_6	A_6	A_6	$0,1,6$
2			$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	H
3				$2,3,4,5$	$2,3,4,5$	$2,3,4,5$	H
4					$2,3,4,5$	$2,3,4,5$	H
5						$2,3,4,5$	H
6							6

From this one obtains: $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = 0,223$, $\widetilde{\mu}_2(2) = \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = \widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) = 0,267$. Therefore we have $\forall\, r > 2$, $^rH = {}^2H$ and $s.f.g.(H)$=2.

c_{31})

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			2	5	$0,1$	6	3
3				$0,1$	6	2	4
4					2	3	5
5						4	$0,1$
6							2

The corresponding membership function is: $\widetilde{\mu}(0) = \widetilde{\mu}(1) = 0,643$, $\widetilde{\mu}(2) = \widetilde{\mu}(3) = \widetilde{\mu}(4) = \widetilde{\mu}(5) = \widetilde{\mu}(6) = 1$ and the associated join space is

^1H	0	1	2	3	4	5	6
0	$0,1$	$0,1$	H	H	H	H	H
1		$0,1$	H	H	H	H	H
2			$2,3,4,5,6$	$2,3,4,5,6$	$2,3,4,5,6$	$2,3,4,5,6$	$2,3,4,5,6$
3				$2,3,4,5,6$	$2,3,4,5,6$	$2,3,4,5,6$	$2,3,4,5,6$
4					$2,3,4,5,6$	$2,3,4,5,6$	$2,3,4,5,6$
5						$2,3,4,5,6$	$2,3,4,5,6$
6							$2,3,4,5,6$

One obtains: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,202$, $\widetilde{\mu}_1(2)=\widetilde{\mu}_1(3)=\widetilde{\mu}_1(4)=\widetilde{\mu}_1(5)=\widetilde{\mu}_1(6)=0,175$. By consequence, $\forall\, r > 1$, ${}^rH = {}^1H$.

c$_{32}$)

H	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		0	2	3	4	5	6
2			4	0, 1	3	5	6
3				4	2	5	6
4					0, 1	5	6
5						0, 1, 2, 3, 4	6
6							A_6

We have: $\widetilde{\mu}(0)=\widetilde{\mu}(1) = 0,552$, $\widetilde{\mu}(2)=\widetilde{\mu}(3)=\widetilde{\mu}(4)=0,796$, $\widetilde{\mu}(5)=0,924$, $\widetilde{\mu}(6)=1$. From $\widetilde{\mu}$ one obtains the join space

${}^1H$	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2, 3, 4	0, 1, 2, 3, 4	0, 1, 2, 3, 4	A_6	H
1		0, 1	0, 1, 2, 3, 4	0, 1, 2, 3, 4	0, 1, 2, 3, 4	A_6	H
2			2, 3, 4	2, 3, 4	2, 3, 4	2, 3, 4, 5	2, 3, 4, 5, 6
3				2, 3, 4	2, 3, 4	2, 3, 4, 5	2, 3, 4, 5, 6
4					2, 3, 4	2, 3, 4, 5	2, 3, 4, 5, 6
5						5	5, 6
6							6

The membership function determined by ${}^1H$ is: $\widetilde{\mu}_1(0) = \widetilde{\mu}_1(1) = 0,235$, $\widetilde{\mu}_1(2) = \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) = 0,228$, $\widetilde{\mu}_1(5) = 0,258$, $\widetilde{\mu}_1(6) = 0,290$. Hence we form the join space:

${}^2H$	0	1	2	3	4	5	6
0	0, 1	0, 1	0, 1, 2, 3, 4	0, 1, 2, 3, 4	0, 1, 2, 3, 4	0, 1, 5	0, 1, 5, 6
1		0, 1	0, 1, 2, 3, 4	0, 1, 2, 3, 4	0, 1, 2, 3, 4	0, 1, 5	0, 1, 5, 6
2			2, 3, 4	2, 3, 4	2, 3, 4	A_6	H
3				2, 3, 4	2, 3, 4	A_6	H
4					2, 3, 4	A_6	H
5						5	5, 6
6							6

and we find: $\widetilde{\mu}_2(0) = \widetilde{\mu}_2(1) = 0,239$, $\widetilde{\mu}_2(2) = \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4) = 0,220$, $\widetilde{\mu}_2(5) = 0,269$, $\widetilde{\mu}_2(6) = 0,297$. Therefore we have $\forall\, r > 2$, ${}^rH = {}^2H$ and $s.f.g.(H) = 2$.

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