Some fully nonlinear elliptic boundary value problems with ellipsoidal free boundaries

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In a seminal paper in 1971, J. Serrin [19] showed that the only bounded domain $\Omega \subset \mathbb{R}^N (N \geq 2)$, with $\partial\Omega \in C^2$, on which the solution to the N-dimensional torsion problem ($\Delta u = 1$ in Ω, and $u = 0$ on $\partial\Omega$) has a constant outward normal derivative on $\partial\Omega$, is the N-dimensional ball. Serrin's proof is based on the moving plane method and Hopf's maximum principles. Under weaker assumptions on the regularity of $\partial\Omega$, H.F. Weinberger presented in a subsequent paper [20] a simpler proof of Serrin's basic result. His method of proof combines a maximum principle for a suitable P-function and a Rellich type identity resulting from Green's theorem. However, Weinberger's proof relies on the linearity of the Laplacian, while Serrin's method could be extended to more general nonlinear elliptic equations.

Since then, many authors extended the methods of J. Serrin and H.F. Weinberger to a wide range of overdetermined problems as well as other techniques have been introduced to obtain analogous results for yet other overdetermined problems. Despite the fact that nowadays there is a vast literature on overdetermined problems for partial differential equations, the overwhelming majority of these problems deals with overspecified data which force the underlying domain to be a ball (see e.g. the survey paper of P.W. Schaefer [18] and its references). Therefore, the literature on characterizations of other shapes, such as, e.g. N-dimensional ellipsoids, in terms of overdetermined problems is still relatively little. This thing is mainly due to the fact that the classical methods of investigation employed to characterize balls, such as, e.g. moving plane method or maximum principle approach for "standard" P-functions, cannot be applied to characterize ellipsoids.

In this talk we are going to investigate three classes of overdetermined boundary value problems for fully nonlinear elliptic equations. In each case, it will be shown that if a solution exists, then the underlying domain must be the interior of an ellipsoid (or ellipse in two dimensions).

Assume throughout that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded C^2 domain, with $\mathbf{0} \in \Omega$. We consider the following type of boundary value problems:

$$\begin{cases} F(D^2 u) = c & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $c > 0$ is a constant, D^2u denotes the Hessian matrix of $u(\mathbf{x})$, $F : S^N(\mathbb{R}) \to \mathbb{R}$ is a C^1 convex function with $F(O) = 0$ and $S^N(\mathbb{R})$ denotes the space of $N \times N$ real symmetric matrices. Moreover, the nonlinearity F is supposed to satisfy *the uniform ellipticity condition*:

(E) There exists $\lambda > 0$ such that

$$F_{ij}(M)\xi_i\xi_j \geq \lambda |\xi|^2 \quad \text{for all } (M, \xi) \in S^N \times \mathbb{R}^N, \tag{2}$$

where $F_{ij}(D^2u) := \partial F / \partial u_{,ij}(D^2u)$.

Under these assumptions, equations of the form (1) are called *fully nonlinear second order uniformly elliptic equations*. Notice that the class of partial differential equations (1) includes various well-known fully nonlinear elliptic equations, such as, e.g. Pucci equations, Bellman equations, Isaacs equations or Hessian equations (see L.A. Caffarelli and X. Cabre [2], p.17-18).

For some particular choices of the nonlinearity F in (1), it is not difficult to verify that if Ω is an ellipsoid then problem (1) admits a unique solution which satisfies some geometric properties. A natural question which arises is to determine which of these properties are also necessary, i.e. which overdetermined conditions on the problems (1) can force the domain Ω to be the interior of an N-dimensional ellipsoid. Our talk will provide some answers in this direction.

The main results of this talk are included in [2]. We list them below

1. The general case: $F = F(D^2u)$**.**

Theorem 1. *Let u be the solution of the problem (1). Suppose that an interior level set of u is homothetic to $\partial\Omega$. Then Ω must be an ellipsoid.*

2. Functions of the eigenvalues of the Hessian: $F = f(\lambda_1, ..., \lambda_n)$.

Let h be the support function of Ω, i.e. the distance from the tangent plane of $\partial\Omega$ to the origin.

Theorem 2. *Let u be an admissible solution of the problem (1). If u satisfies the further boundary condition*

$$|\nabla u| = \frac{1}{h}, \text{on } \partial\Omega, \tag{3}$$

then Ω must be an ellipsoid.

3. Hessian equations: $F = S_k(\lambda_{,1}, ..., \lambda_n)$.

Let $k = (k_1, ..., k_{n-1})$ the principal curvatures of $\partial\Omega$ and $G = k_1...k_{n-1}$.be the Gauss curvature of $\partial\Omega$. We have:

Theorem 3. *Let u be an admissible solution of the problem (1).If u satisfies the further boundary condition*

$$G\,|\nabla u|^n = h, \text{ on } \partial\Omega, \tag{4}$$

then Ω must be an ellipsoid.

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