Some remarks on the ideal of bilinear nuclear operators

GABRIELA BADEA

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In this presentation we study the bilinear nuclear operators and some of their properties. Thus, we prove that the set of bilinear nuclear operators is a normed ideal of operators and it is the smallest one among all other ideals. We also prove a result that gives a necessary and sufficient condition for an operator on $c_0 \times c_0$ to be multiple summing.

Key words: bilinear nuclear operators, multiple summing operators

1. INTRODUCTION AND NOTATION

In [5] M.Matos establishes the definition of multilinear mapping of nuclear type which is the natural generalization of the nuclear linear operator. In a second paper, [6], the same author continues to study this type of operators and extends even more the concept of linear nuclear operator to the virtually nuclear multilinear mappings.

In this paper we will study both concepts defined by M.Matos and some of their properties in the bilinear case. We prefer the bilinear case instead of the multilinear one because for $n = 2$, the writting becomes a lot easier.

Now we fix our notation and then we recall some definitions.

For X, Y, Z Banach spaces, we denote by $\mathcal{L}(X, Y; Z)$ the bilinear continuous mappings from $X \times Y$ to Z, with the bilinear operatorial norm

$$\|U\|^{bil} = \sup_{\|x\|\leq 1, \|y\|\leq 1} \|U(x, y)\| .$$

For X Banach space, X^* will be the topological dual of X and B_X the closed unit ball of X.

Next we recall the definitions for both concepts of nuclear operator in the bilinear case, which are the extensions of the linear nuclear operator. These concepts were given by M.Matos (see [5],[6]).

Definition 1. *A mapping $U \in \mathcal{L}(X, Y; Z)$ is of nuclear type if it has a nuclear representation of the form*

$$U = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n^* \otimes z_n, \tag{1}$$

i.e. for every $(x,y) \in X \times Y$ *we have*

$$U(x,y) = \sum_{n=1}^{\infty} \lambda_n x_n^*(x)\, y_n^*(y)\, z_n, \tag{2}$$

with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, $\sup_{n\in\mathbb{N}} \|x_n^*\| < \infty, \sup_{n\in\mathbb{N}} \|y_n^*\| < \infty, \sup_{n\in\mathbb{N}} \|z_n\| < \infty.$

The vector space of all such mappings is denoted by $\mathcal{N}(X,Y;Z)$ and we consider the nuclear norm

$$\|U\|_{nuc} = \inf \sum_{n=1}^{\infty} \|\lambda_n\| \cdot \|x_n^*\| \cdot \|y_n^*\| \cdot \|z_n\|$$

where the infimum is taken over all possible representations of U as described in (1).

Next we have the definition of the virtually nuclear operator, also given by M.Matos in [6].

Definition 2. *A mapping* $U \in \mathcal{L}(X,Y;Z)$ *is called virtually nuclear if there exist sequences* $(\lambda_{nm})_{n,m\in\mathbb{N}} \subset \mathbb{K}, (x_n^*)_{n\in\mathbb{N}} \subset X^*, (y_m^*)_{m\in\mathbb{N}} \subset Y^*, (z_{nm})_{n,m\in\mathbb{N}} \subset Z$ *such that* $\sum_{n,m=1}^{\infty} \|x_n^*\| \, \|y_m^*\| \, \|z_{nm}\| < \infty$ *and for every* $(x,y) \in X \times Y$,

$$U(x,y) = \sum_{n,m=1}^{\infty} \lambda_{nm} x_n^*(x)\, y_m^*(y)\, z_{nm}, \tag{3}$$

with $\sum_{n,m=1}^{\infty} |\lambda_{nm}| < \infty$, $\sup_{n\in\mathbb{N}} \|x_n^*\| < \infty, \sup_{m\in\mathbb{N}} \|y_m^*\| < \infty, \sup_{n,m\in\mathbb{N}} \|z_{nm}\| < \infty.$

The vector space of all such mappings is denoted by $\mathcal{N}_{virt}(X,Y;Z)$ and we consider the virtually nuclear norm of U

$$\|U\|_{virt} = \inf \sum_{n,m=1}^{\infty} \|\lambda_{nm}\| \cdot \|x_n^*\| \cdot \|y_m^*\| \cdot \|z_{nm}\|$$

where the infimum is taken over all possible representations of U as described in (3).

We also nead the definition of bilinear multiple p-summing operators. This type of summability was defined and studied by M.Matos in [6] and independently by F.Bombal, D.Perez-Garcia, I. Villanueva in the article [2].

For $1 \le p < \infty$ and every $x_1, ..., x_n \in X$ we denote $w_p(x_i \mid 1 \le i \le n) = \sup \left\{ \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{\frac{1}{p}} \mid x^* \in B_{X^*} \right\}.$

Definition 3. *Let $1 \leq p < \infty$. A bilinear operator $U \in \mathcal{L}(X,Y;Z)$ is multiple p-summing if there exists a constant $C > 0$ such that for every choice of elements $x_1,...,x_k \in X, y_1,...,y_l \in Y$ we have the following relation*

$$\left(\sum_{i,j=1}^{k,l} \|U(x_i,y_j)\|^p\right)^{\frac{1}{p}} \leq C \cdot w_p(x_i \mid 1 \leq i \leq k) \cdot w_p(y_j \mid 1 \leq j \leq l). \tag{4}$$

The vector space of all such mappings is denoted by $\Pi_p^{mult}(X,Y;Z)$ and the multiple p-summing norm of U is defined by $\pi_p^{mult}(U) = \inf\{C > 0 \mid C \text{ verifies } (4)\}$.

Remark 1. *Let $U \in \mathcal{L}(X,Y;Z)$ a multiple p-summing operator. Then for every choice of elements $x_1,...,x_k \in X, y_1,...,y_l \in Y$ we have*

$$\left(\sum_{i,j=1}^{k,l} \|U(x_i,y_j)\|^p\right)^{\frac{1}{p}} \leq \pi_p^{mult}(U) \cdot w_p(x_i \mid 1 \leq i \leq k) \cdot w_p(y_j \mid 1 \leq j \leq l). \tag{5}$$

Proof. Indeed, using the definition of the multiple p-summing norm, we have that there exists a sequence $(C_k)_{k\in\mathbb{N}} \subset \{C > 0 \mid C \text{ verifies } (4)\}$ such that $\lim_{k\to\infty} C_k = \pi_p^{mult}(U)$. If we write the relation (4) for every $C_k, k \in \mathbb{N}$ then

$$\left(\sum_{i,j=1}^{k,l} \|U(x_i,y_j)\|^p\right)^{\frac{1}{p}} \leq C_k \cdot w_p(x_i \mid 1 \leq i \leq k) \cdot w_p(y_j \mid 1 \leq j \leq l)$$

and for $k \to \infty$, the wanted relation is proved. □

In the paper [4], K.Floret and D.Garcia, following A. Pietsch, [7], give an explicit definition for ideals of multilinear operators, which we will next recall in the bilinear case.

Definition 4. *A subclass $\mathcal{A}$ of the class $\mathcal{L}$ of all bilinear and continuous mappings between Banach spaces is called an ideal of bilinear mappings if*
i) For all Banach spaces X_1, X_2, Y the component $\mathcal{A}(X_1,X_2;Y) := \mathcal{L}(X_1,X_2;Y) \cap \mathcal{A}$ is a linear subspace of $\mathcal{L}(X_1,X_2;Y)$.
ii) If $T_i \in \mathcal{L}(X_i,Y_i), \varphi \in \mathcal{A}(Y_1,Y_2;Z)$ and $S \in \mathcal{L}(Z,W)$ then $S \circ \varphi \circ (T_1,T_2) \in \mathcal{A}$.
iii) The mapping $I : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $I(\lambda_1,\lambda_2) = \lambda_1\lambda_2, I \in \mathcal{A}$.

Definition 5. *A normed ideal is a pair $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ where $\mathcal{A}$ is an ideal and $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to [0,\infty)$ is an ideal norm i.e.*
i) $\|\cdot\|_{\mathcal{A}}$ restricted to each component is a norm.
ii) $\|S \circ \varphi \circ (T_1,T_2)\|_{\mathcal{A}} \leq \|S\| \cdot \|\varphi\|_{\mathcal{A}} \cdot \|T_1\| \cdot \|T_2\|$ in the situation ii) from the previous definition.
iii) $\|I\|_{\mathcal{A}} = 1$.

2. THE RESULTS

We start with some results which appear in the remark 4.3 in Matos's paper [6] without a proof.

Proposition 6. *i) The set $\mathcal{N}(X,Y;Z)$ is a linear subspace in the vector space $\mathcal{L}(X,Y;Z)$ and we have*

$$\|U\|^{bil} \leq \|U\|_{nuc} \qquad (U \in \mathcal{N}(X,Y;Z)) \tag{6}$$

ii) $\mathcal{N}(X,Y;Z) = \mathcal{N}_{virt}(X,Y;Z)$ isometrically.

Now we give an exemple of bilinear nuclear operator.

Example 7. *Let $x^* \in X^*, y^* \in Y^*$ and $z \in Z$. Then $U : X \times Y \to Z$, given by $U = x^* \otimes y^* \otimes z$, i.e. $U(x,y) = x^*(x)\, y^*(y)\, z, (x,y) \in X \times Y$ is a nuclear operator and $\|U\|_{nuc} = \|x^*\| \|y^*\| \|z\|$.*

Our next result proves that in the case of bilinear nuclear operator, the ideal property also holds .

Theorem 8. *i) Let $X_i, Y_i, i = 1,2$ and Z, Z_1 be Banach spaces. Let $u_i : X_i \to Y_i$ linear and continuous mappings and $S : Y_1 \times Y_2 \to Z$ a bilinear nuclear operator. Then $T : X_1 \times X_2 \to Z, T = S \circ (u_1, u_2)$ is a bilinear nuclear operator and*

$$\|T\|_{nuc} \leq \|S\|_{nuc} \|u_1\| \|u_2\|.$$

ii) Let $T : X_1 \times X_2 \to Z$ be a bilinear nuclear operator and $V : Z \to Z_1$ a linear and continuous one, then $V \circ T : X_1 \times X_2 \to Z_1$ is a bilinear nuclear operator and $\|V \circ T\|_{nuc} \leq \|V\| \|T\|_{nuc}$.

Since $\mathcal{N}(X,Y;Z)$ is a linear subspace (proposition 6) and it has the ideal propertie (theorem 8), using the definitions 4 and 5, we can state the following corollary.

Corollary 9. *The set $\mathcal{N}(X,Y;Z)$ with the nuclear norm is a normed ideal of bilinear operators.*

Next we have that the ideal of bilinear nuclear operators is the smallest ideal among all other ideals of bilinear operators.

Theorem 10. *Let $\mathcal{U}(X,Y;Z)$ be an ideal of bilinear operators and $\mathcal{A}$ the norm defined on $\mathcal{U}$. Then $\mathcal{N}(X,Y;Z) \subset \mathcal{U}(X,Y;Z)$ and $\mathcal{A}(U) \leq \|U\|_{nuc}$, for any choice of $U \in \mathcal{N}(X,Y;Z)$.*

In the followings we will study a particular type of bilinear operator, that is the operator defined on $c_0 \times c_0$. The next result gives a neccesary and a sufficient condition for an operator to be multiple 1-summing. It also proves that in the same hypothesis, the virtually nuclear bilinear operator coincides with the multiple 1-summing one, that is $\mathcal{N}_{virt}(c_0, c_0; X) = \Pi_1^{mult}(c_0, c_0; X)$.

Theorem 11. *Let $U : c_0 \times c_0 \to X$ a bilinear and continuous operator. Then the following statements are equivalent:*
i) U is a multiple 1-summing operator
ii) The serie $\sum_{n,m=1}^{\infty} \| U(e_n, e_m) \| < \infty$, where $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of c_0.
iii) U is a virtually nuclear operator.
Moreover, we have

$$\pi_1^{mult}(U) = \sum_{n,m=1}^{\infty} \| U(e_n, e_m) \| .$$

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